What's new in Normaliz?

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Universität Osnabrück

What's Normaliz?

NORMALIZ

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open source

Linux, Mac & Windows

C++ (boost & GMP)

GUI Interface jNormaliz

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Paralleliza-tion (OpenMP)

PyNormaliz

C++ library libnormaliz

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Version 3.2.0 just released!

http://www.math.uos.de/normaliz

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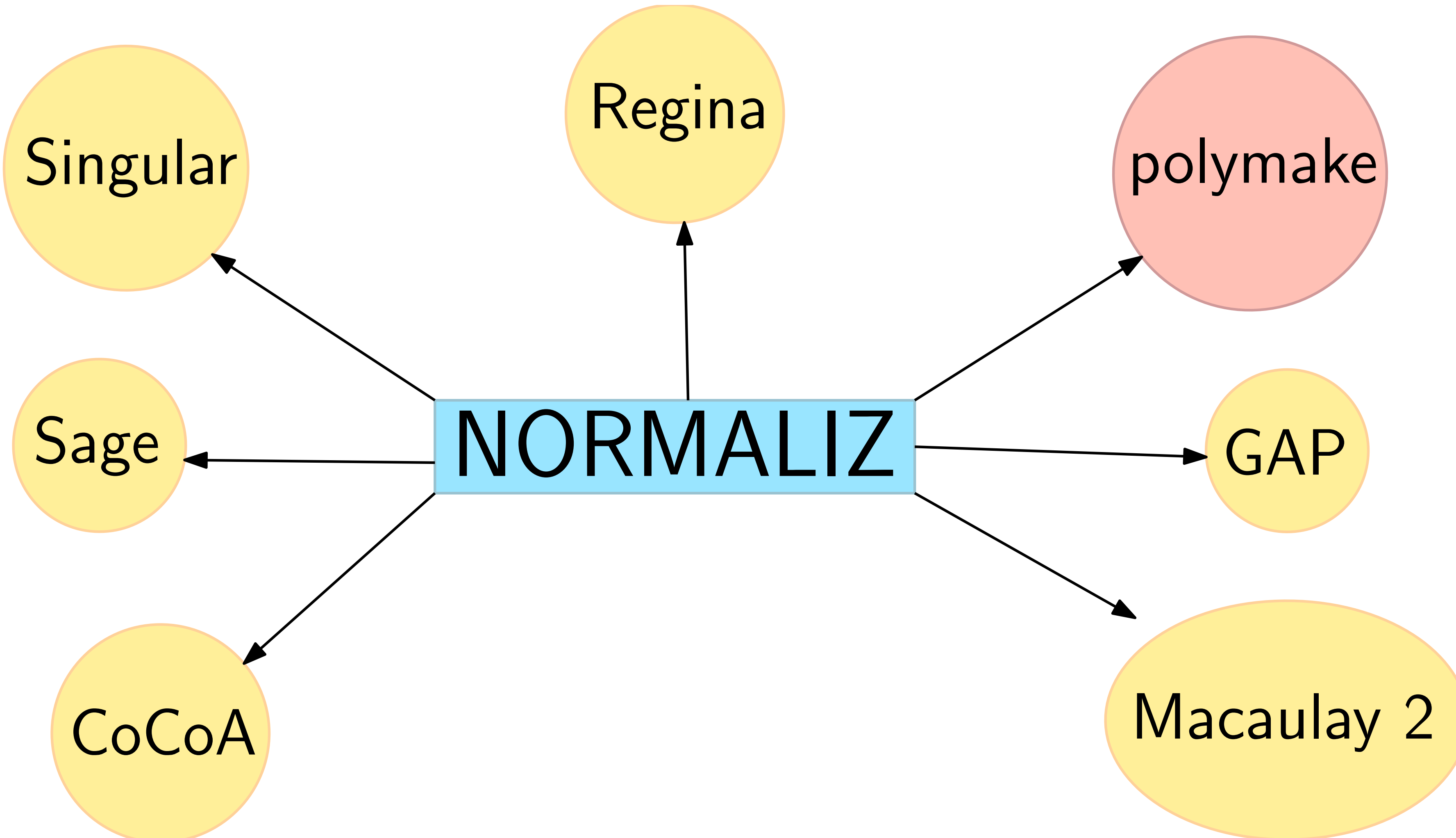

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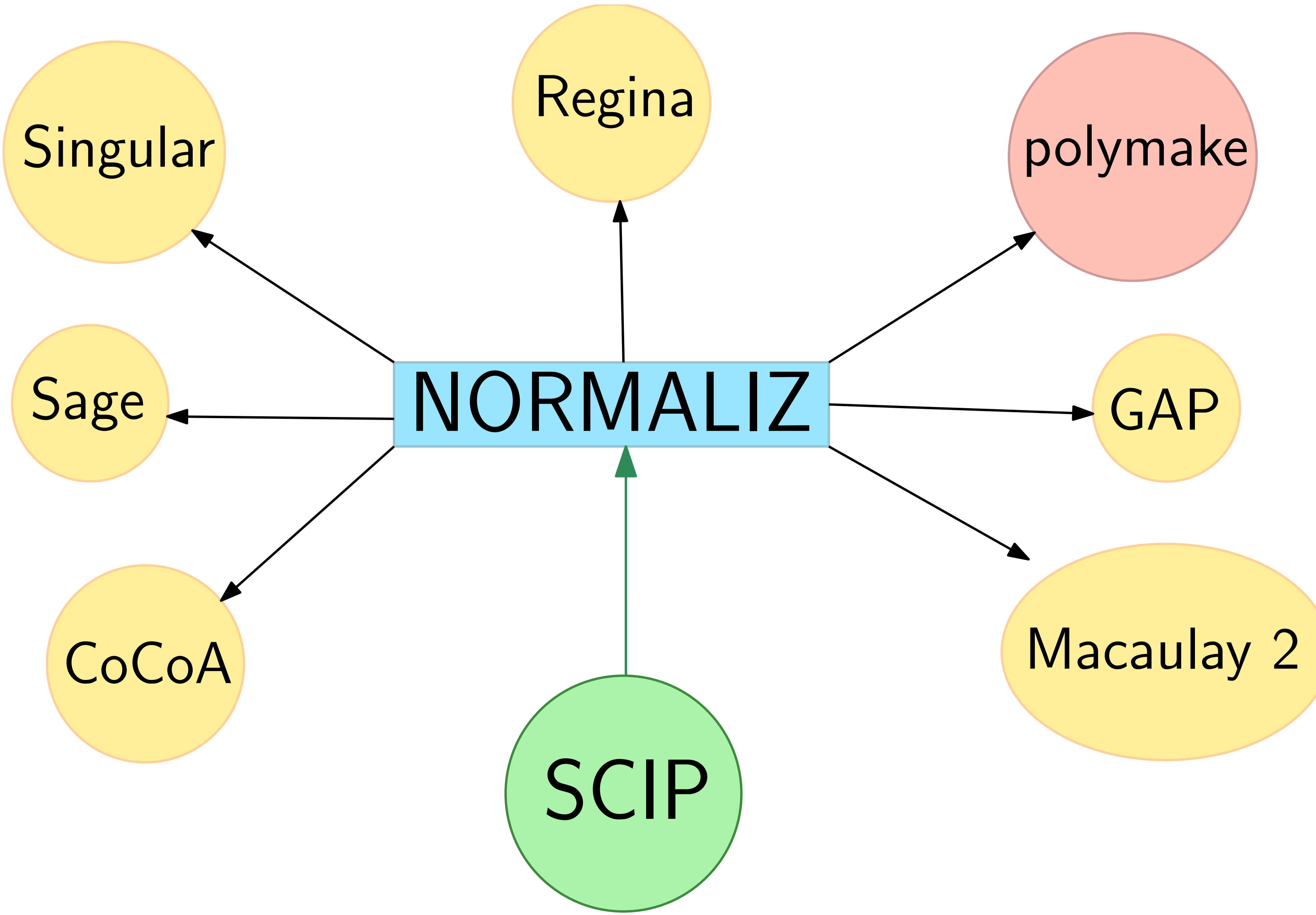
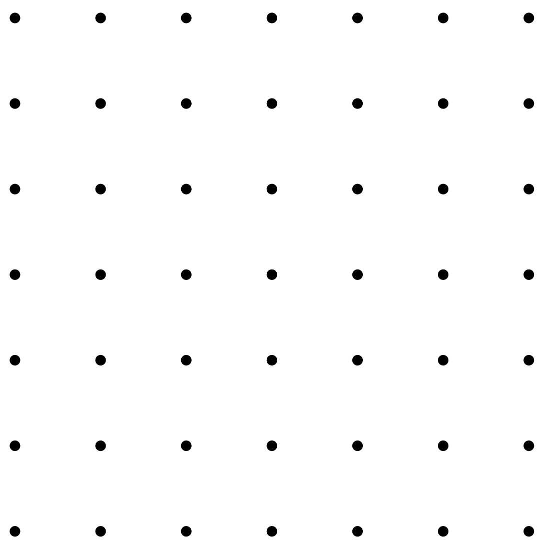


Rational Cones

L . . . a lattice (subgroup of $\mathbb{Z}^d$)



Rational Cones

L . . . a lattice (subgroup of $\mathbb{Z}^d$)

C . . . a (rational polyhedral) cone

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\begin{aligned}
C &= \operatorname{cone}(x_1, \ldots, x_n) \subset \mathbb{R}^d \\
&= \{a_1 x_1 + \cdots + a_n x_n \mid a_1, \ldots, a_n \in \mathbb{R}_+\} \\
&= \{x \in \mathbb{R}^n \mid Ax \geq 0\}
\end{aligned}
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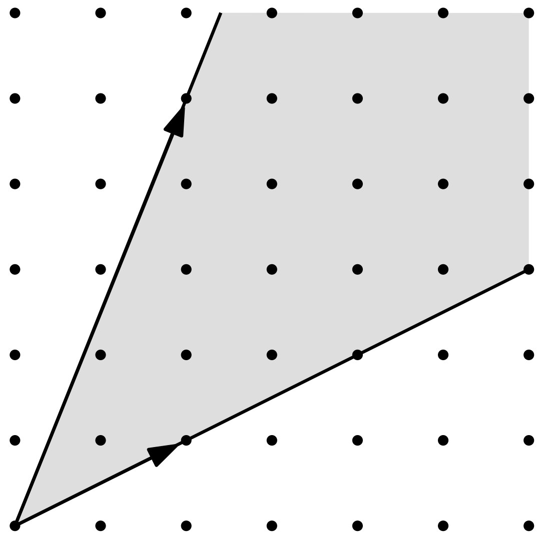
with a generating system $x_1, \ldots, x_n \in \mathbb{Z}^d$.



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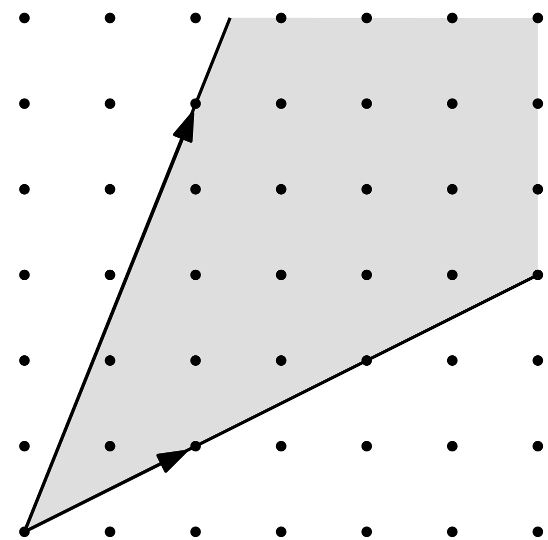

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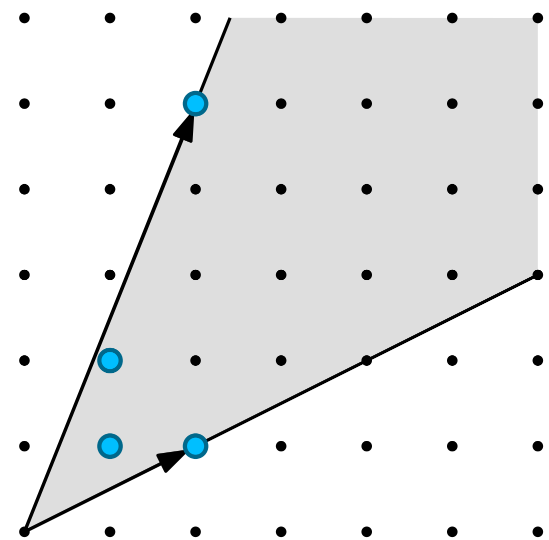

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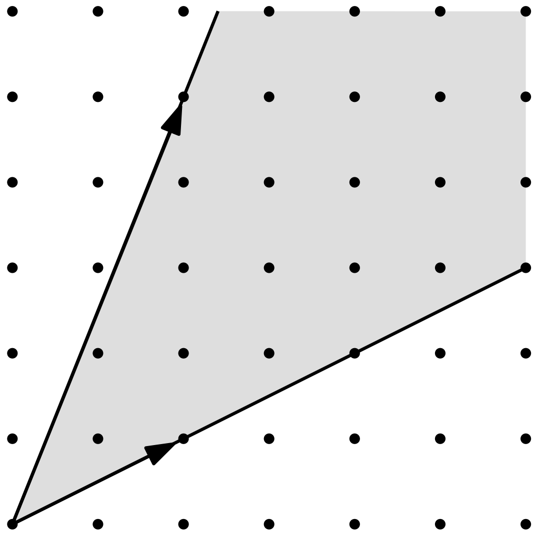
Theorem [Gordan's Lemma]

Let $C \subset \mathbb{R}^d$ be the cone generated by $x_1, \dots, x_n \in \mathbb{Z}^d$. Then $C \cap L$ is an affine monoid M, i.e. a finitely generated submonoid of $\mathbb{Z}^d$.

The Tasks of Normaliz: Hilbert Basis

Assume C pointed: $x, -x \in C \Rightarrow x = 0$.



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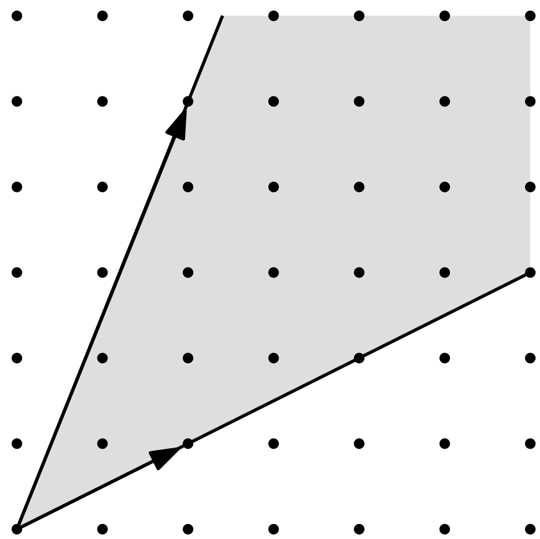

$x \in M = C \cap L, x \neq 0$ is irreducible:

$$x = y + z \;\Rightarrow\; y = 0 \text{ or } z = 0.$$

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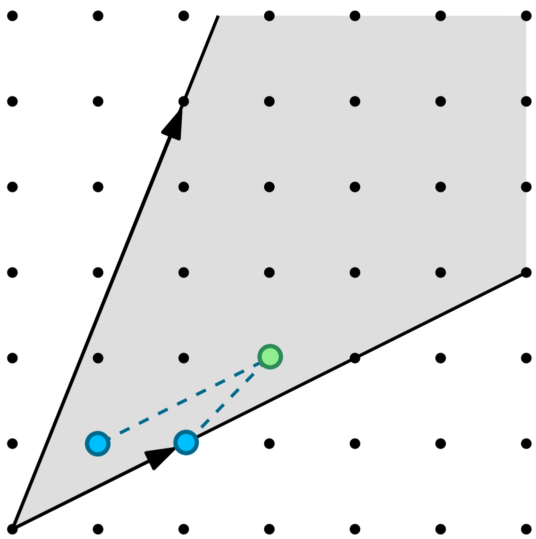

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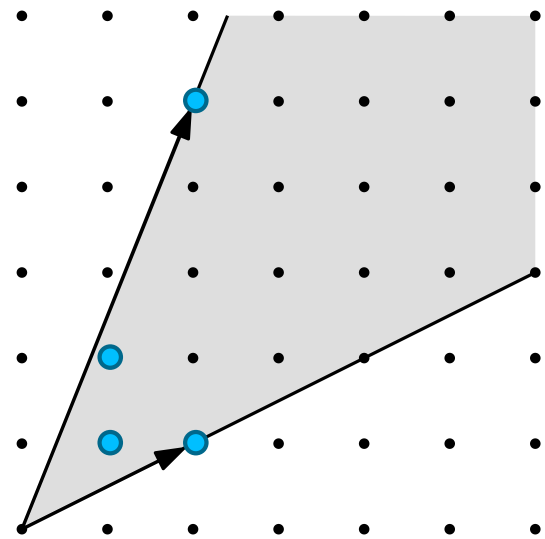


Theorem

There are only finitely many irreducible elements in $C \cap L$ and they form the unique minimal system of generators, the Hilbert Basis.

The Tasks of Normaliz: Hilbert Series

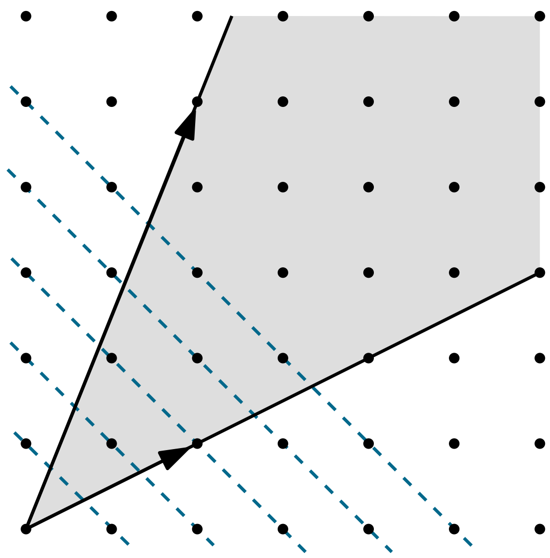
Second main task: Count lattice points by degree

Hilbert (Ehrhart) function

$$H(M,k) = \#\{x \in M \mid \deg x = k\}$$

Hilbert (Ehrhart) series

$$H_M(t) = \sum_{k=0}^{\infty} H(M,k)t^k.$$



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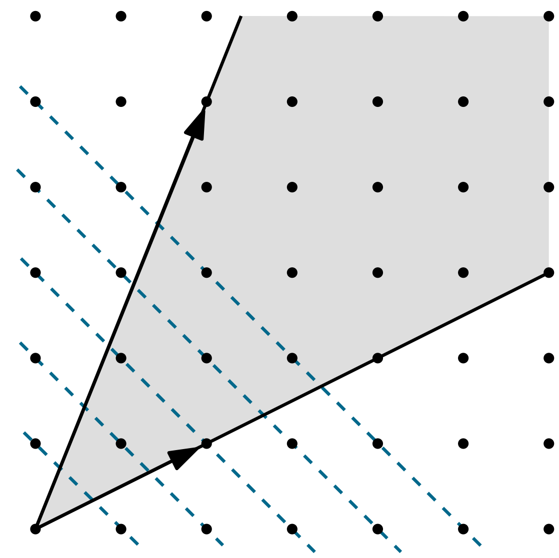

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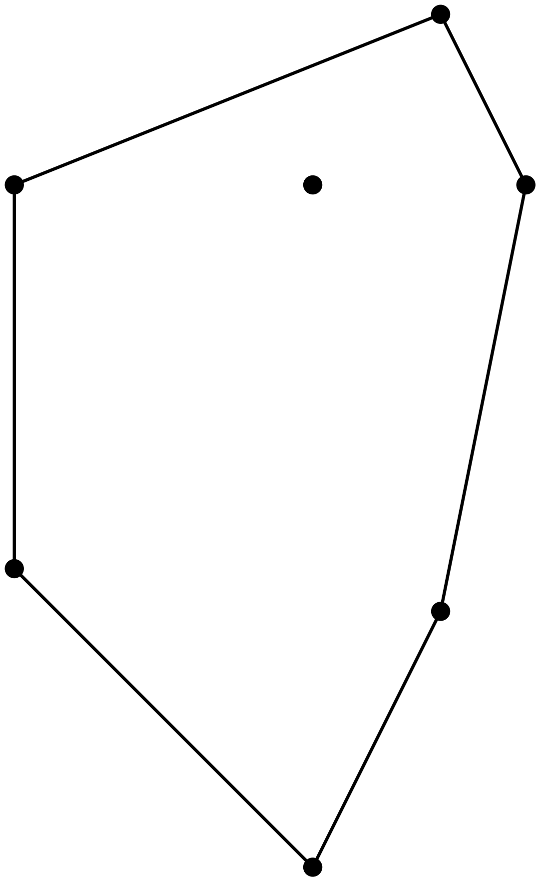
Theorem [Hilbert-Serre, Ehrhart]

- $H_M(t)$ is a rational function.
- $H(M,k)$ is a quasi-polynomial for $k \geq 0$.

Normaliz Algorithm



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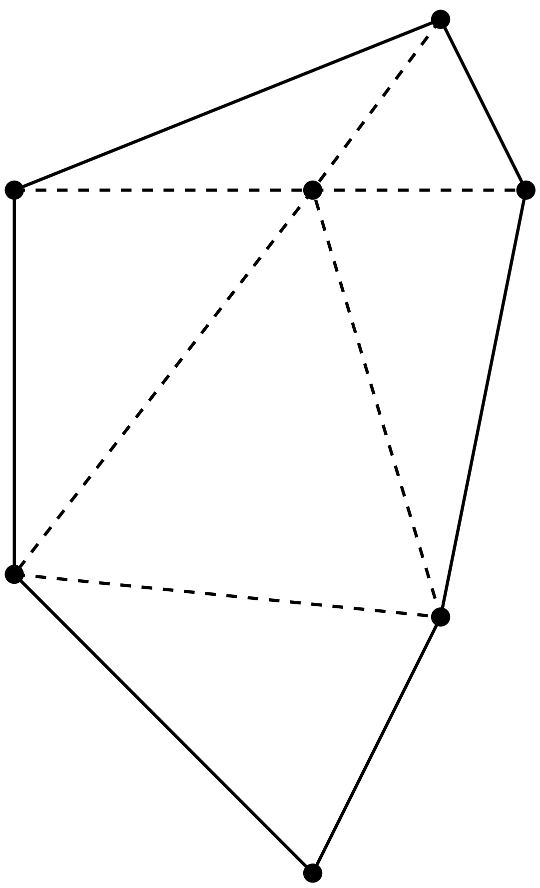

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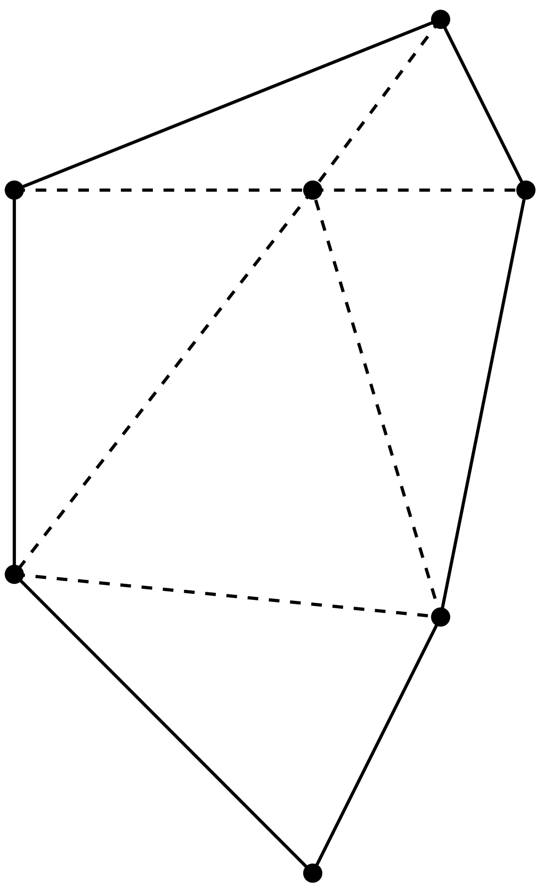

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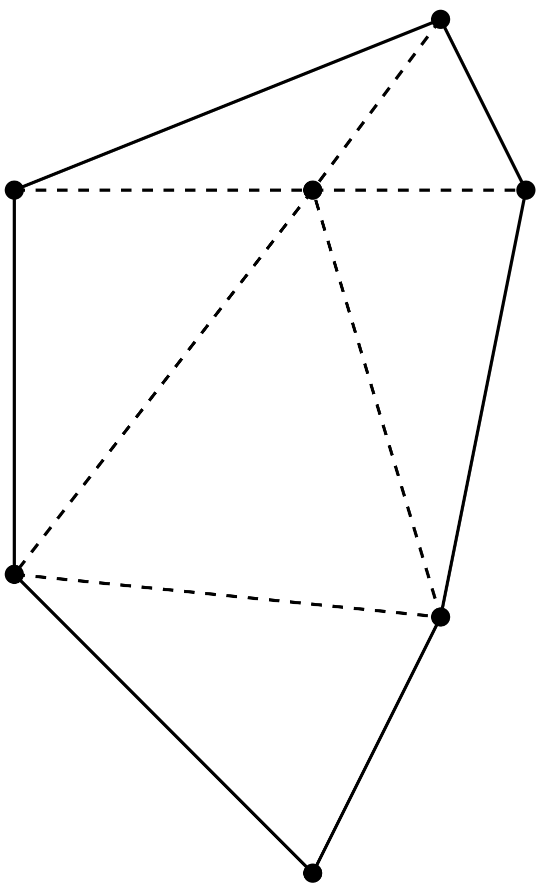

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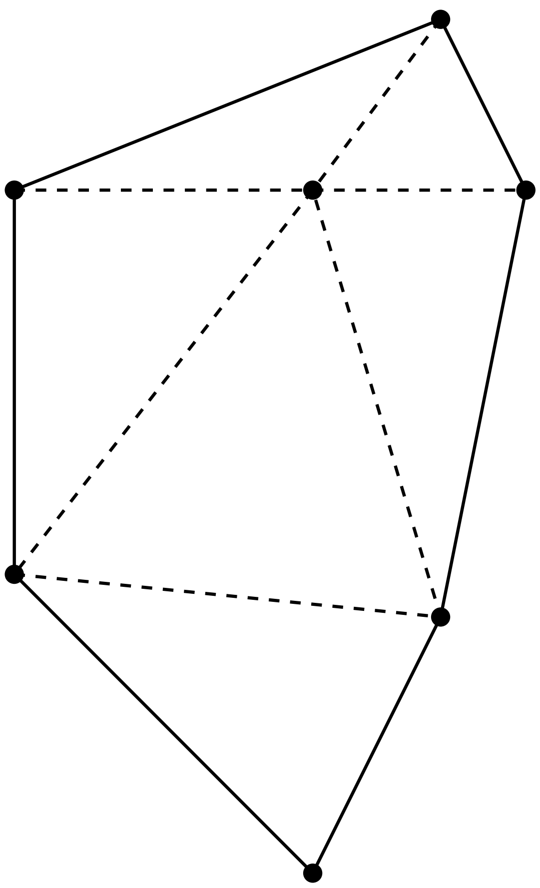

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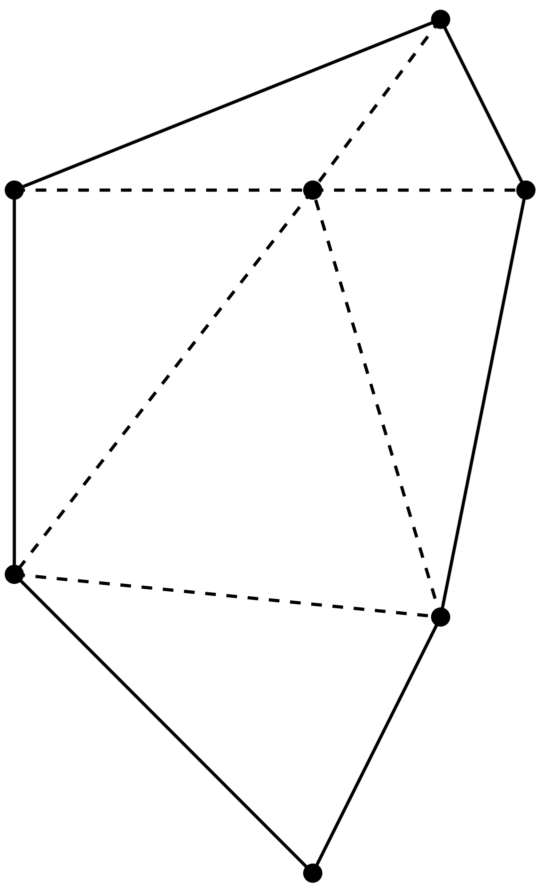

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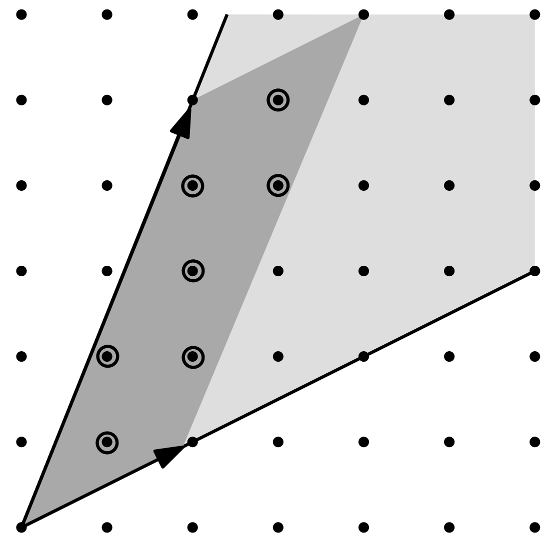
Simplicial Cones

$S = \mathrm{cone}(x_1, \ldots, x_d)$ simplex. Then

$$E = \underbrace{\{q_1 x_1 + \cdots + q_d x_d \mid 0 \le q_i < 1\}}_{\pi} \cap \mathbb{Z}^d$$

together with $x_1, \ldots, x_d$ generate the monoid $S \cap \mathbb{Z}^d$.



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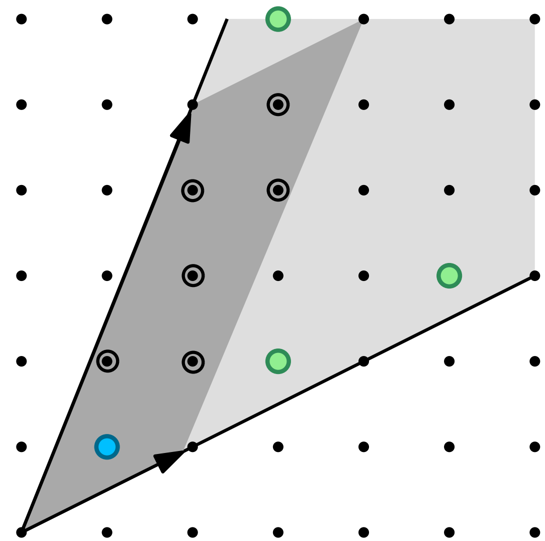

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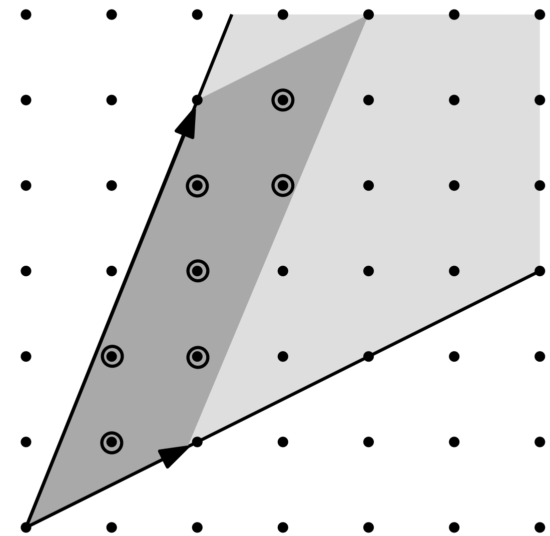

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Normaliz generates the points in E. They are candidates for the Hilbert Basis and their number is given by the volume of the simplex

$$|E| = \mathrm{vol}(S) = \det(x_1, \dots, x_d).$$

The points in E are then reduced to a Hilbert Basis of $S \cap \mathbb{Z}^d$.

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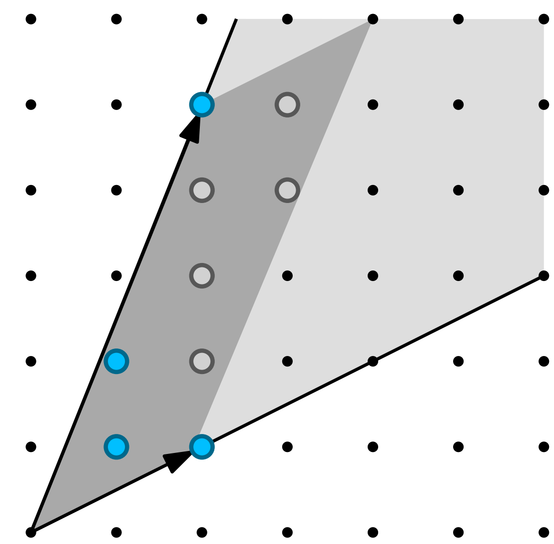

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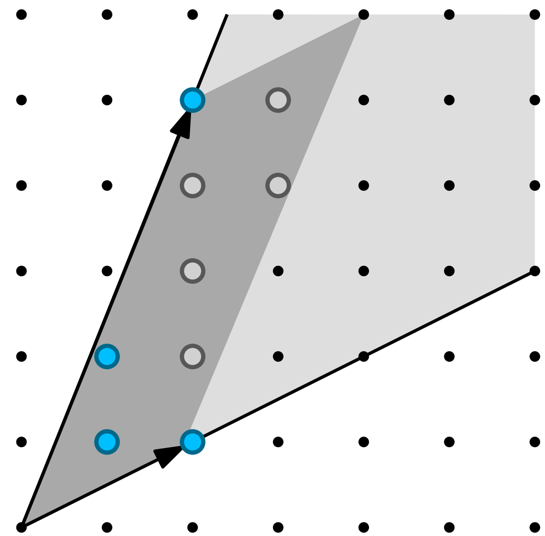

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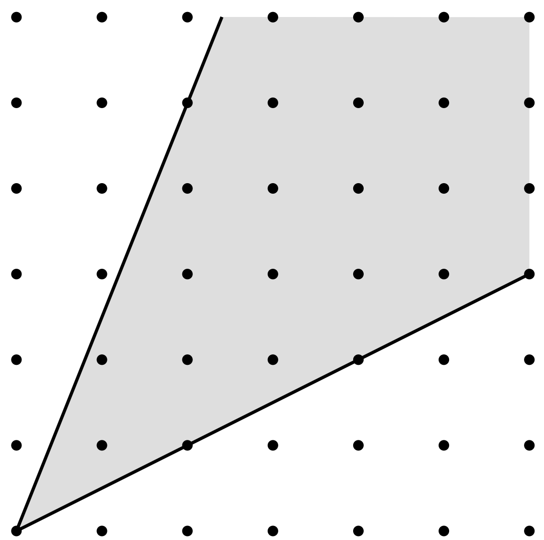
Therefore $\operatorname{vol}(S)$ is a critical size for the runtime of Normaliz.

Our Approach

If simplex S has big volume: decompose it into smaller simplices, such that the sum of their volumes decreases remarkably.



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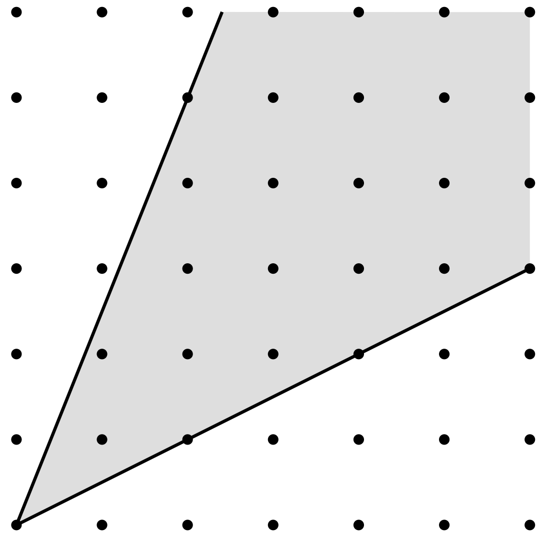

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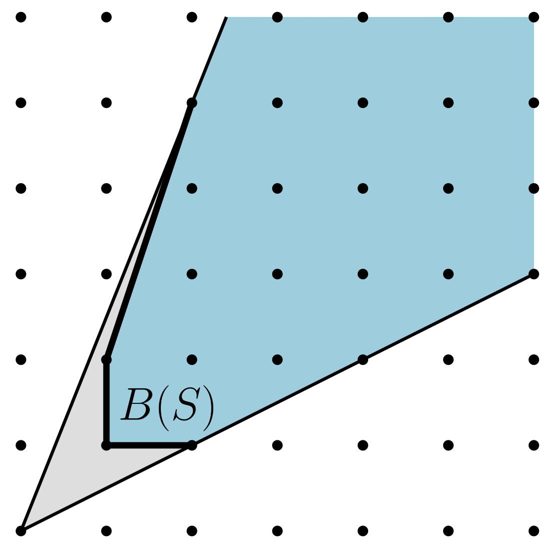

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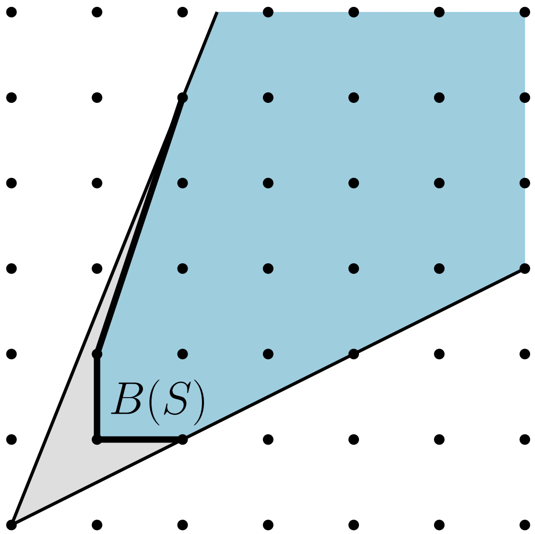

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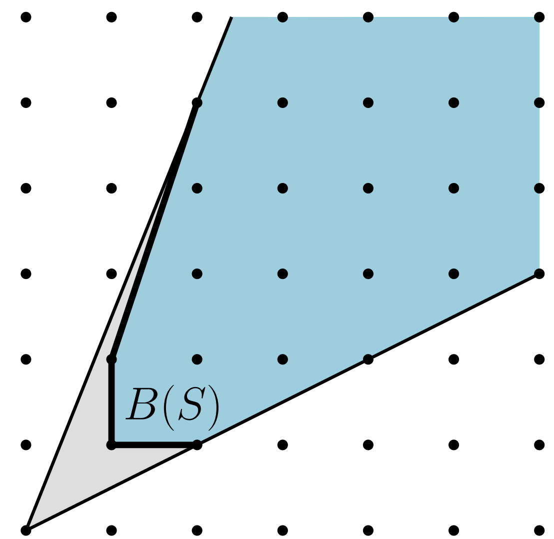

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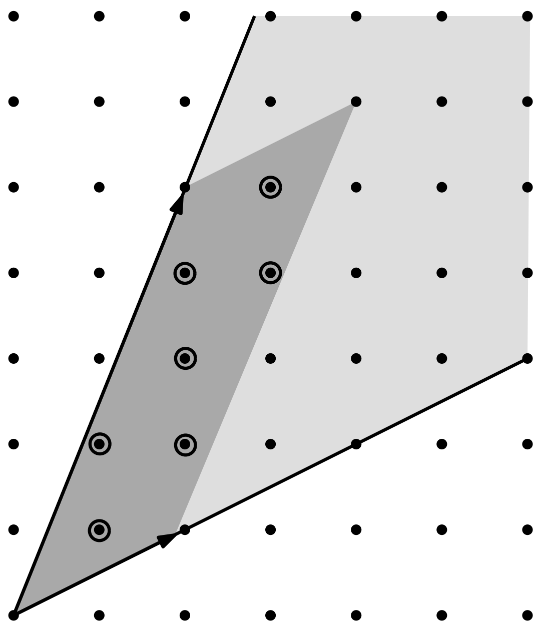
Determine only some points from $B(S)$ using

1. Integer Programming

2. Approximation

Integer Programming

$S = \mathrm{cone}(x_1, \ldots, x_d)$ simplex in triangulation



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Compute a point x that minimizes the sum of determinants:

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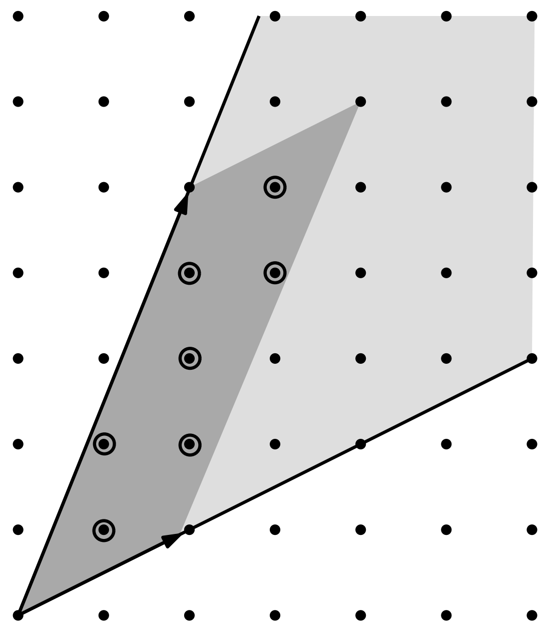
N ... normal vector on the hyperplane spanned by $x_1, \ldots, x_d$.



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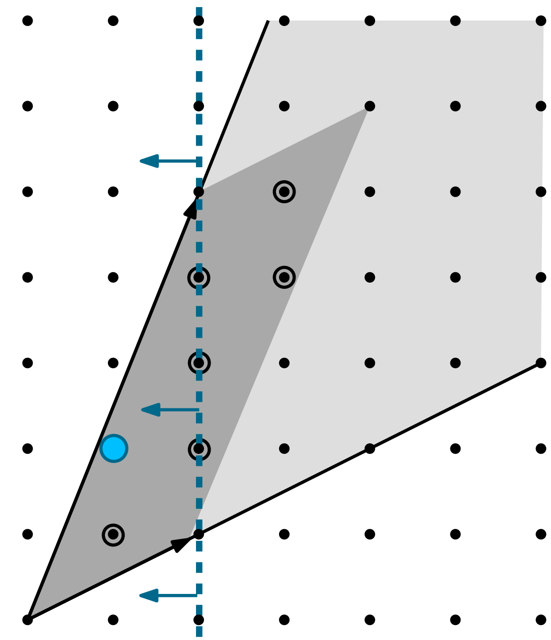

Solve the IP

$$\boxed{\min\{N^T x \mid x \in S \cap \mathbb{Z}^d, x \neq 0, N^T x < N^T x_1\}} \qquad (\star)$$

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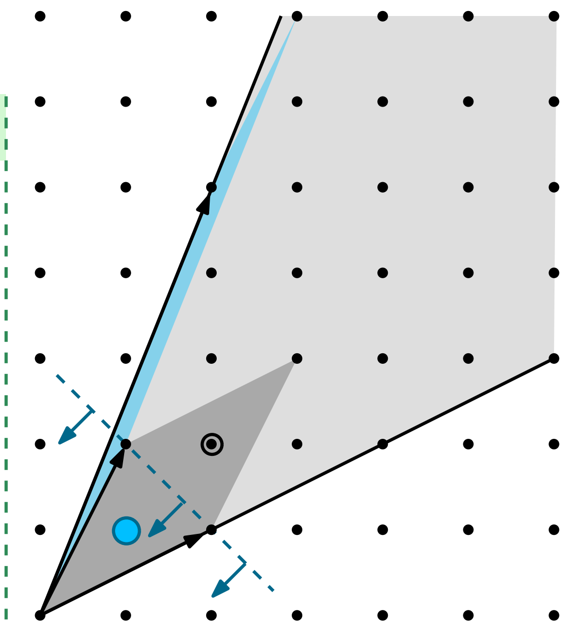

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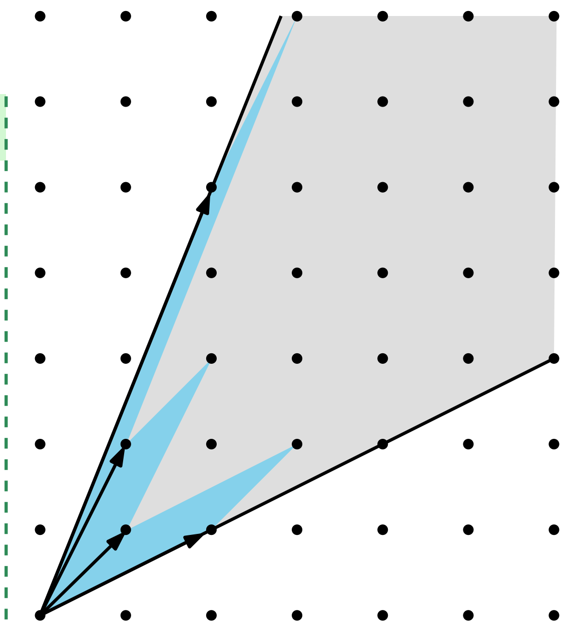

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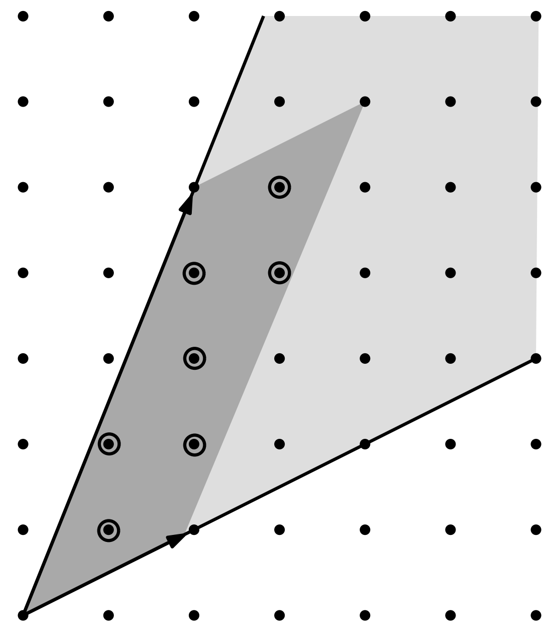
Integer Programming

Algorithm Bottom Points

Input: $S = \mathrm{cone}(x_1, \ldots, x_d)$ simplex with $\mathrm{vol}(S) \geq \mathrm{Bound}$

Return: Points from $B(S)$

1: $\mathcal{B}, \mathcal{S} \leftarrow \emptyset$
2: store S into $\mathcal{S}$
3: **while** $\mathcal{S} \neq \emptyset$ **do**
4: let $T = \mathrm{cone}(y_1, \ldots, y_d)$ be the first element of $\mathcal{S}$ and delete it
5: compute a normal vector N on hyperplane spanned by $y_1, \ldots, y_d$
6: compute hyperplanes $\{H_1, \ldots, H_d\}$ and volume of T
7: **if** $\mathrm{vol}(T) < \mathrm{Bound}$ **then continue**



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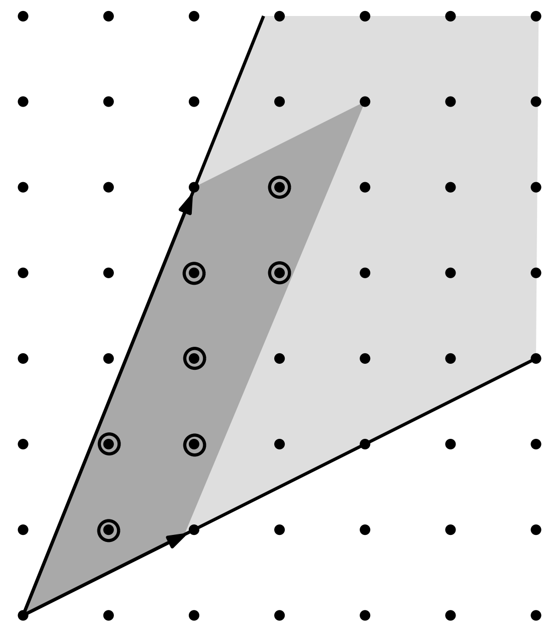


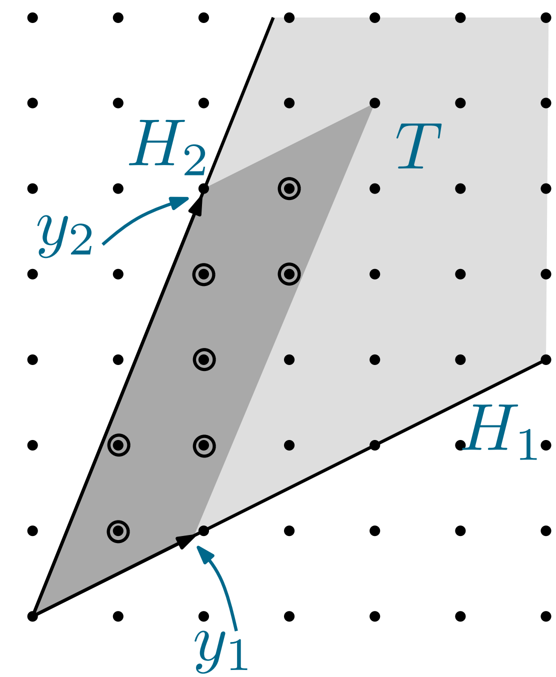
Integer Programming

Algorithm Bottom Points

Input: $S = \mathrm{cone}(x_1, \ldots, x_d)$ simplex with $\mathrm{vol}(S) \geq \mathrm{Bound}$

Return: Points from $B(S)$

$\mathcal{B}, \mathcal{S} \leftarrow \emptyset$
store S into $\mathcal{S}$
while $\mathcal{S} \neq \emptyset$ **do**
    let $T = \mathrm{cone}(y_1, \ldots, y_d)$ be the first element of $\mathcal{S}$ and delete it
    compute a normal vector N on hyperplane spanned by $y_1, \ldots, y_d$
    compute hyperplanes $\{H_1, \ldots, H_d\}$ and volume of T
    if $\mathrm{vol}(T) < \mathrm{Bound}$ **then continue**



Integer Programming

Algorithm Bottom Points

while $\mathcal{S} \neq \emptyset$ **do**

$\vdots$

  if IP $(\star)$ is solvable for T **then**

    $y \leftarrow$ optimal solution of $(\star)$

    store y into $\mathcal{B}$

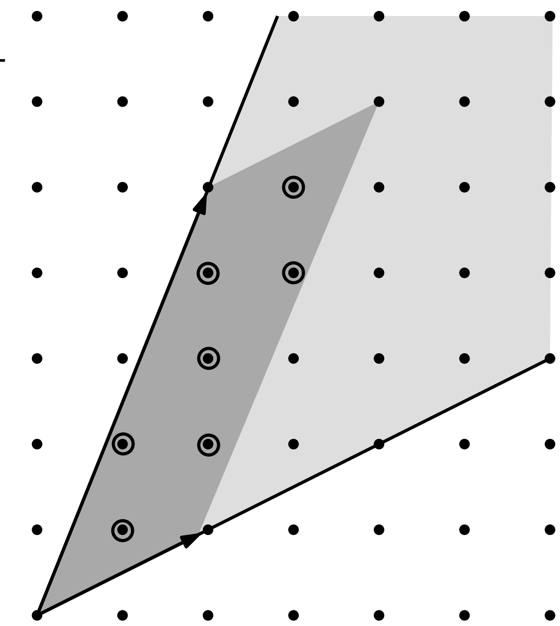
    for all hyperplanes H_i of T **do**

      if $y \notin H_i$ **then**

        $T_i \leftarrow \mathrm{cone}(y_1, \ldots, y_{i-1}, y, y_{i+1}, \ldots, y_d)$

        store T_i into $\mathcal{S}$

return $\mathcal{B}$



$$\boxed{\min\{N^T x \mid x \in S \cap \mathbb{Z}^d, x \neq 0, N^T x < N^T x_1\}} \qquad (\star)$$

Integer Programming

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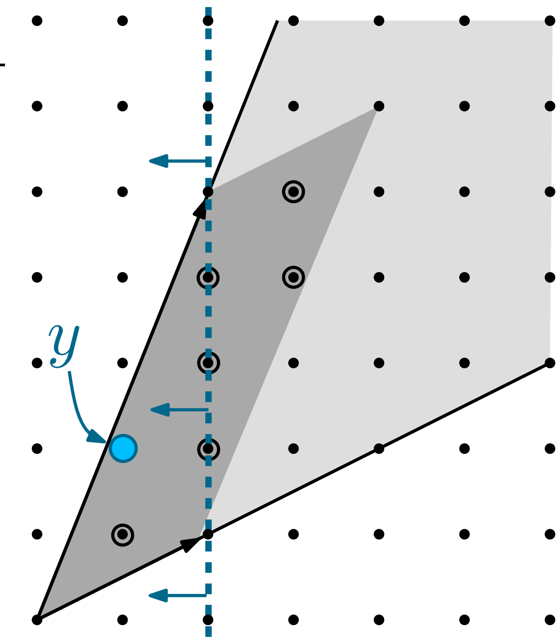

$\mathcal{B} = \{(1,2)$

$$\min\{N^T x \mid x \in S \cap \mathbb{Z}^d, x \neq 0, N^T x < N^T x_1\} \qquad (\star)$$

Integer Programming

Algorithm Bottom Points

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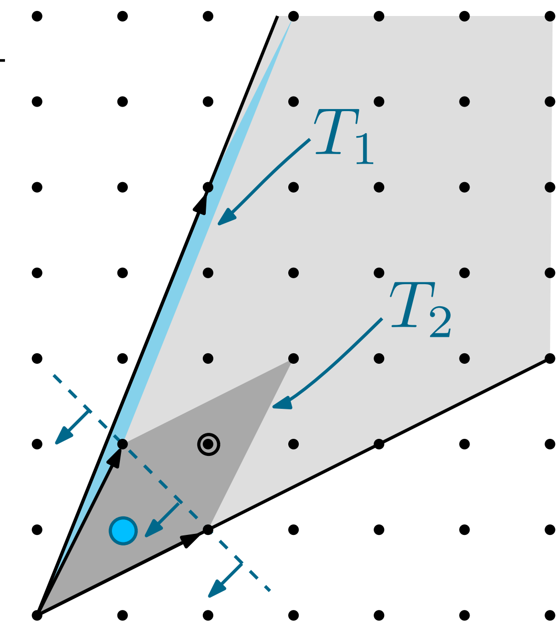

$$\mathcal{B} = \{(1,2),(1,1)\}$$

$$\min\{N^T x \mid x \in S \cap \mathbb{Z}^d, x \neq 0, N^T x < N^T x_1\} \qquad (\star)$$

Integer Programming

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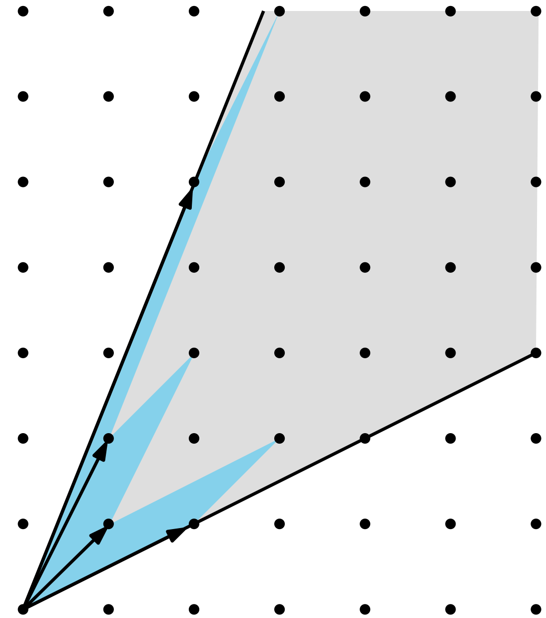

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Integer Programming

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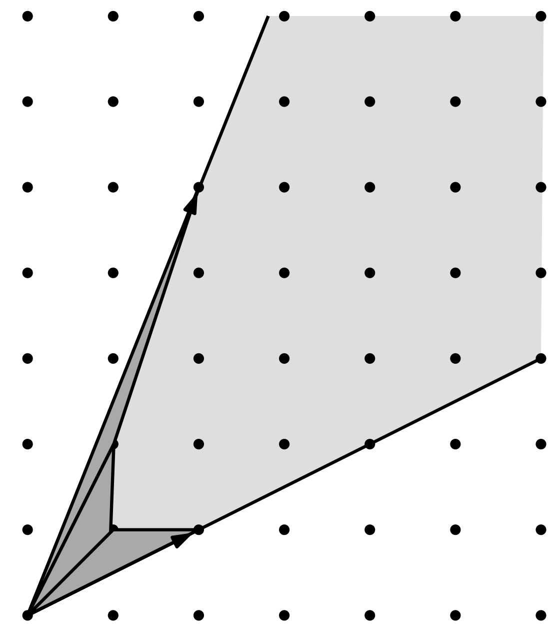
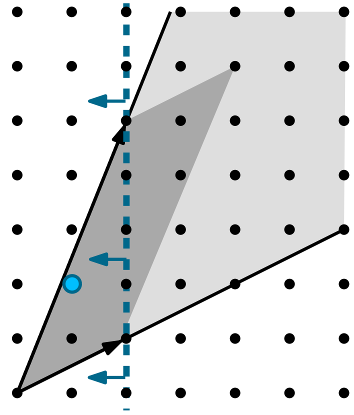


$\mathcal{B} = \{(1,2),(1,1)\}$

$$\min\{N^T x \mid x \in S \cap \mathbb{Z}^d, x \neq 0, N^T x < N^T x_1\} \qquad (\star)$$

We triangulate the lower facets of $\operatorname{conv}(\mathcal{B} \cup \{x_1, \dots, x_d\})$ and evaluate this triangulation with the usual Normaliz algorithm.

Integer Programming

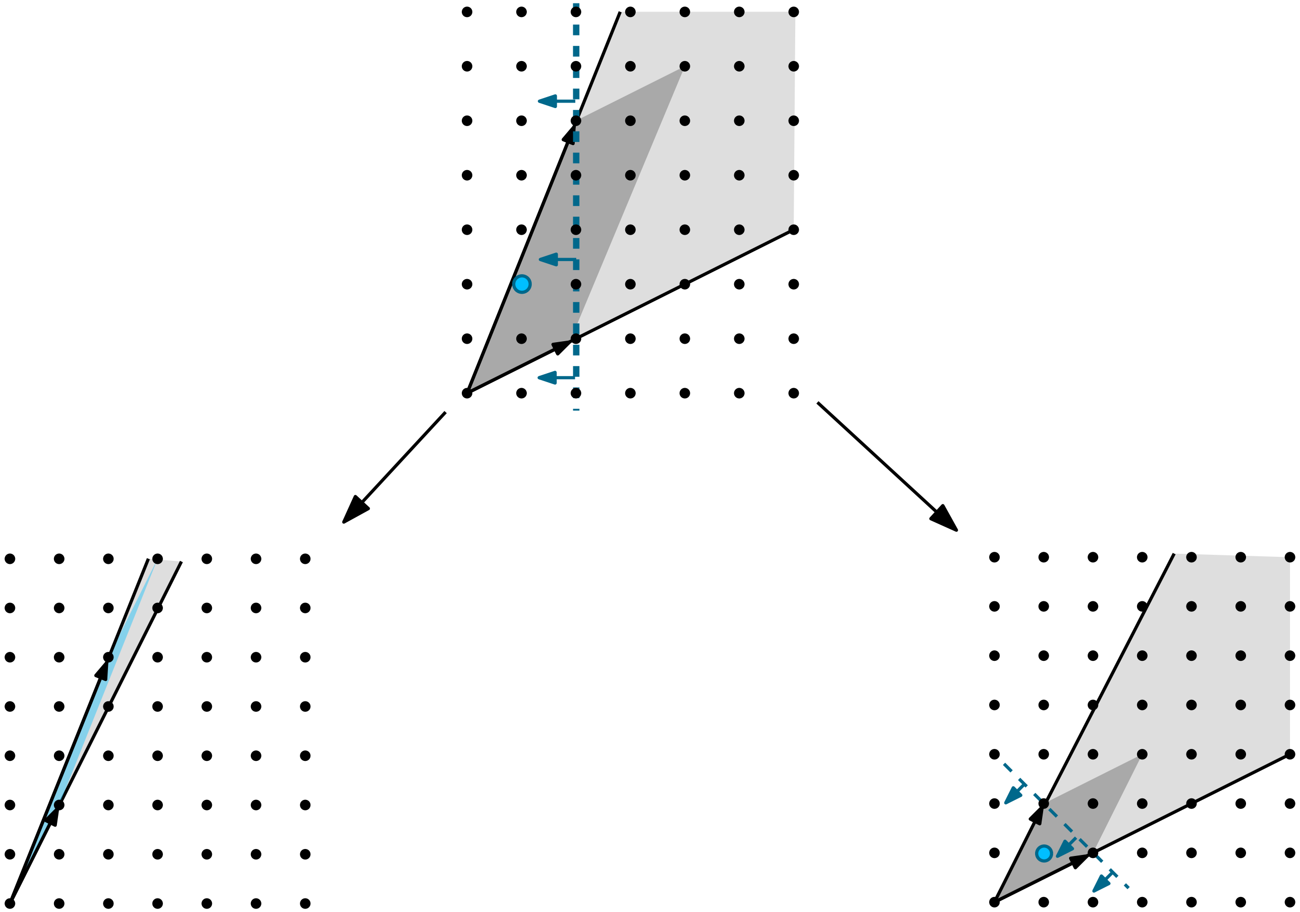


Level 0

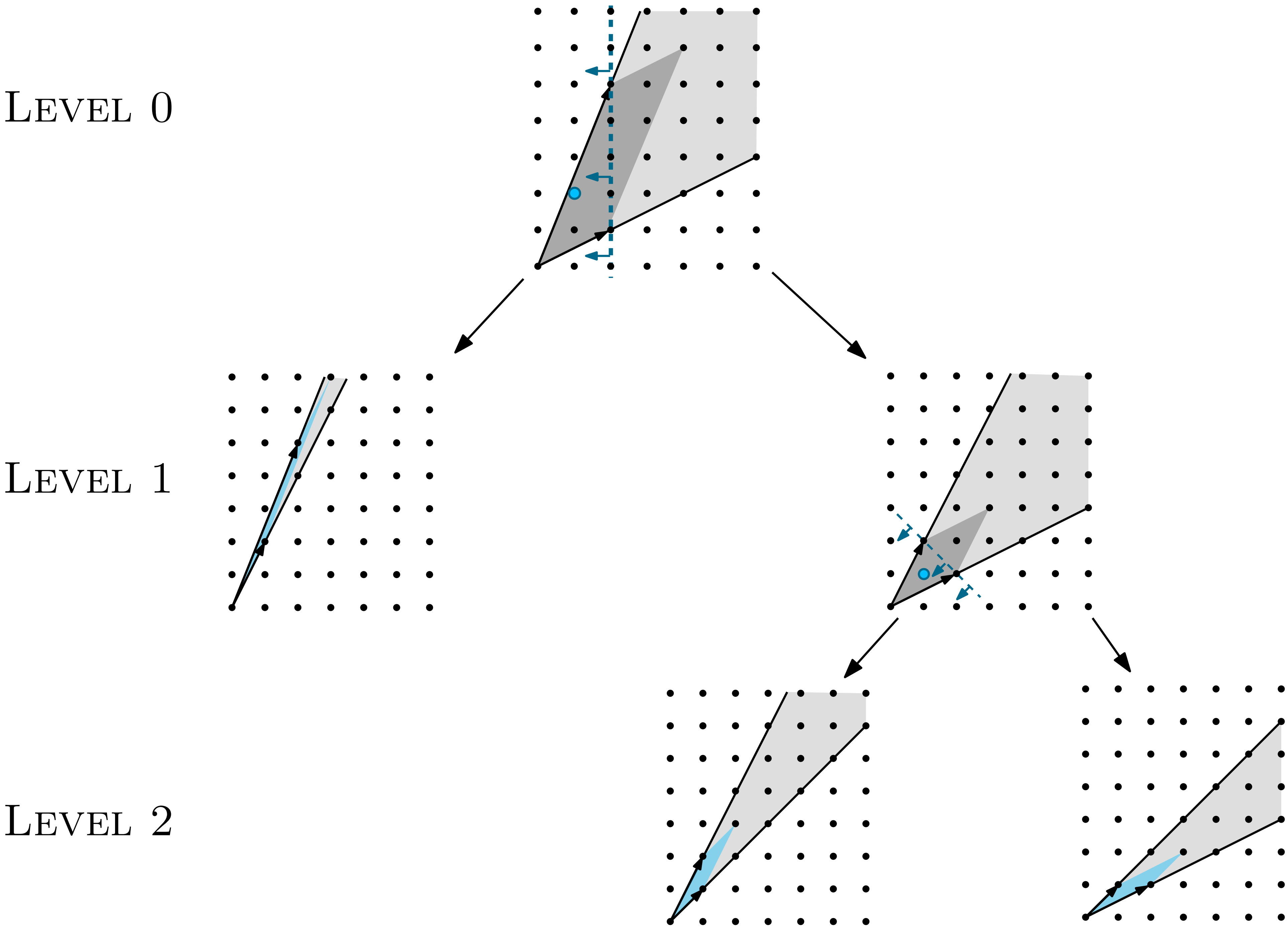
Integer Programming

Level 0

Level 1



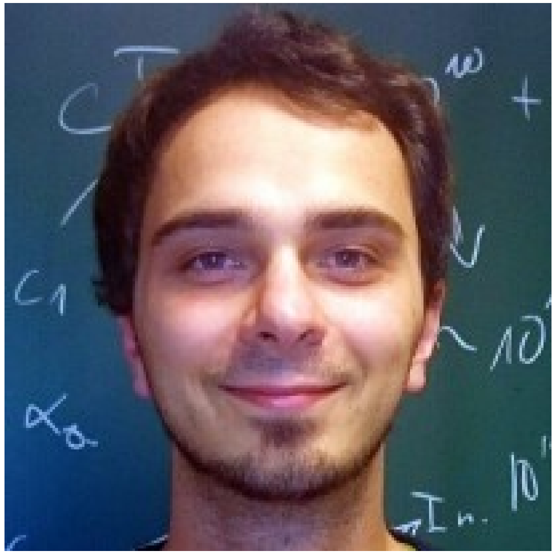
Integer Programming



Implementation & Results

- ⋆ use SCIP (3.2.0) via its C++ interace

ZIB



Gregor Hendel

Implementation & Results

- use SCIP (3.2.0) via its C++ interace
- parallelization with OpenMP
  - individual time limit
  - individual feasibility bounds

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dimension	9	10	12
simplex volume	9.83×10^7	4.17×10^{14}	2.8×10^{14}
bottom volume			
our volume			
improvement factor			
old runtime			
new runtime			

SUN xFire 4450, 4 Intel Xeon X7460 processors, 20 threads, $\text{SCIPBound} = 10^6$

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new runtime	0.5s	36s	4s

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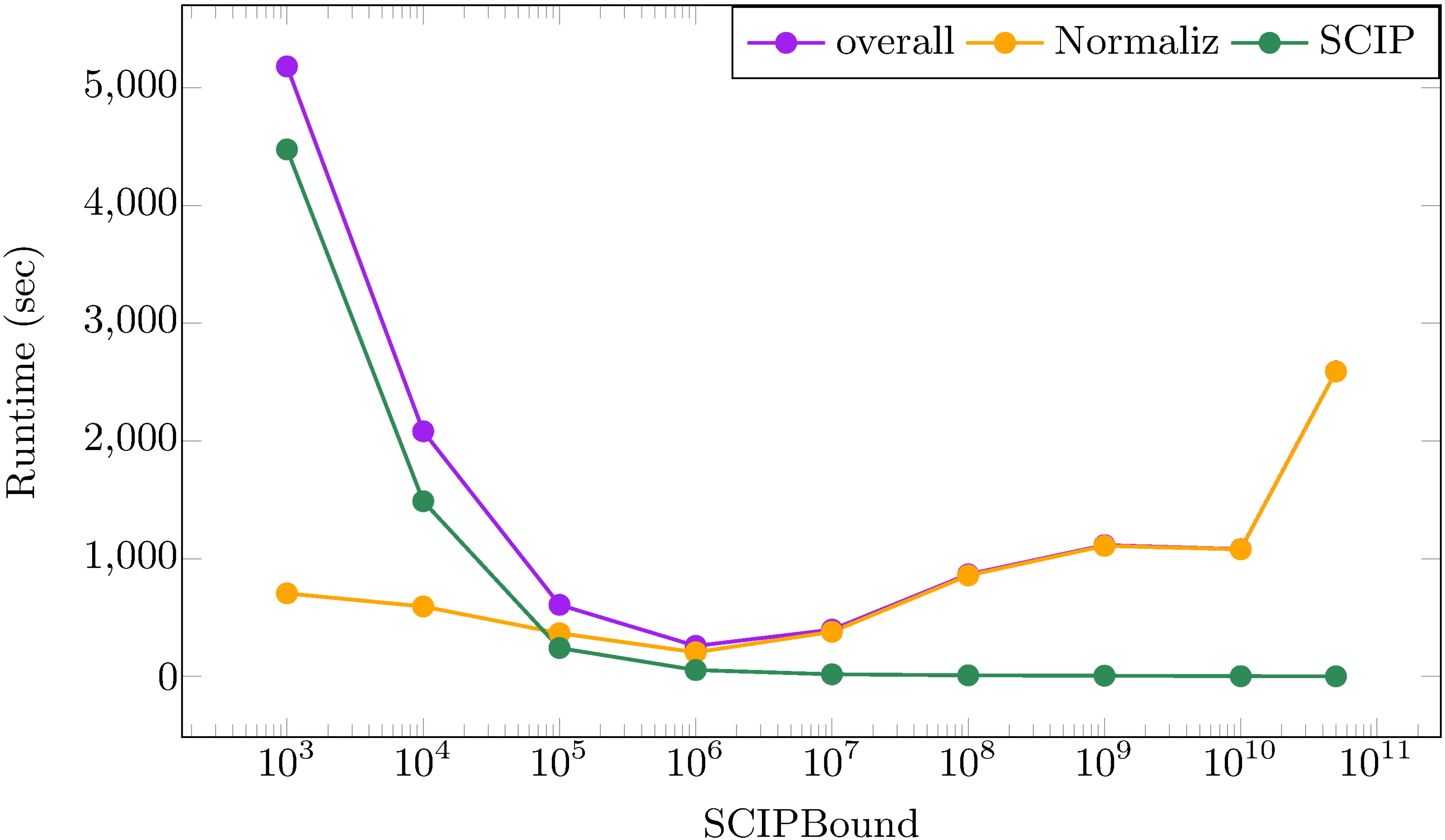
Implementation & Results



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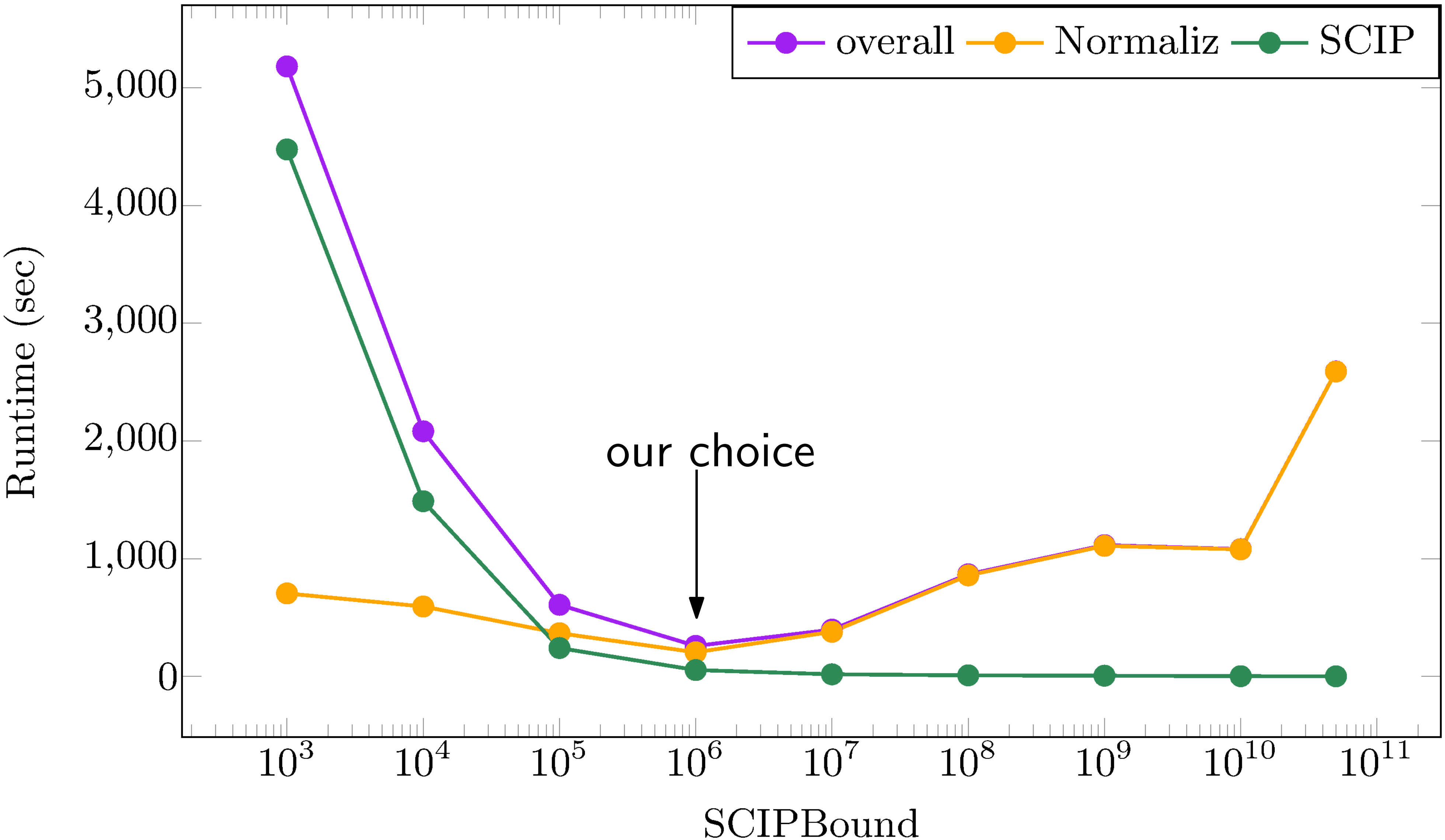
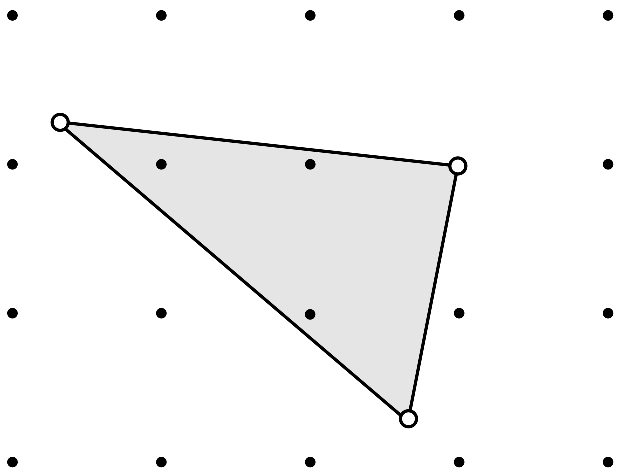


Approximation

1. Look at the cross section at level 1 of the (transformed) simplex.



cross section at level 1

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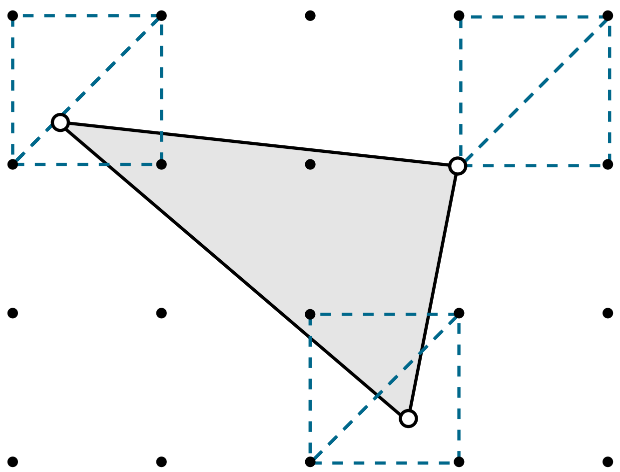
1. Look at the cross section at level 1 of the (transformed) simplex.

2. For each extreme ray/point, triangulate the lattice cube around it using the hyperplane arrangement $\mathcal{A}_n = \{x_i = x_j\}$.



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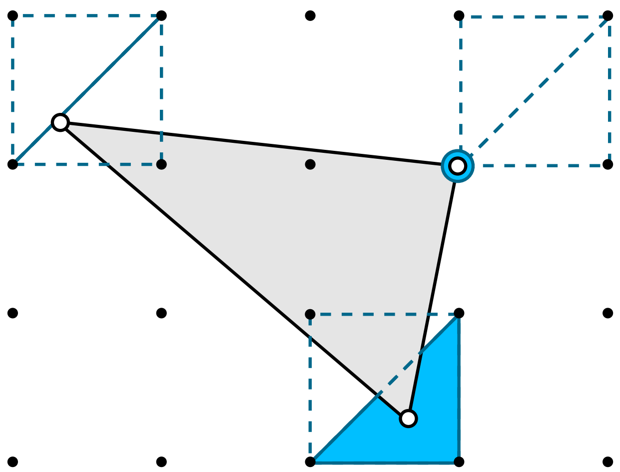
3. Detect the minimal face containing the point and collect its vertices (at most d).



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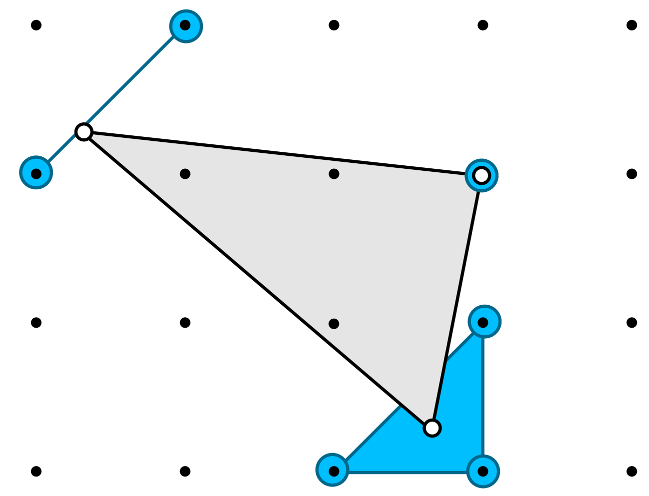

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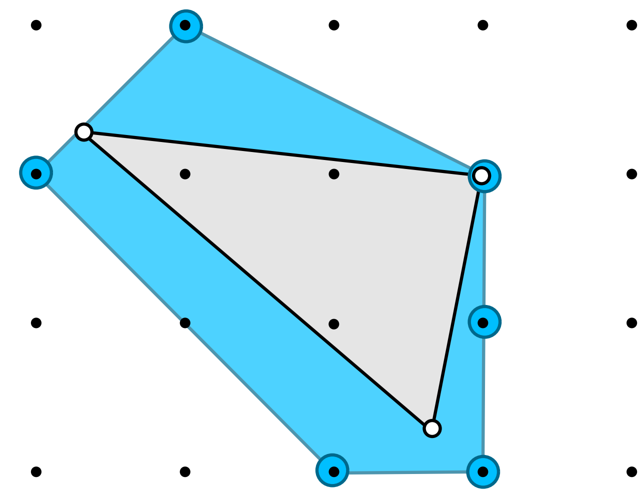
4. Create a candidate list of the new cone, intersect it with the original cone and do local reduction.



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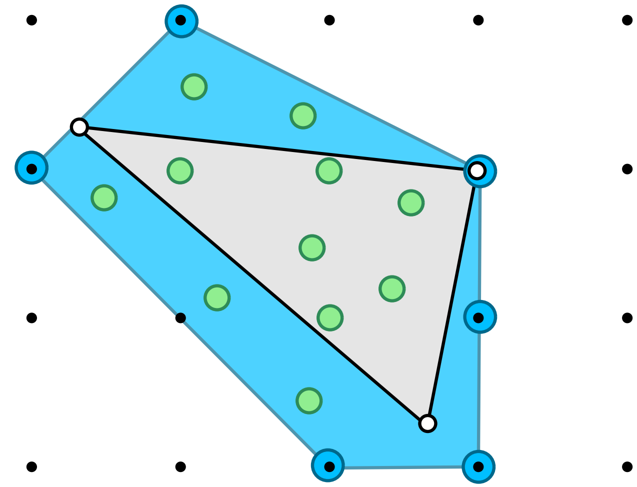

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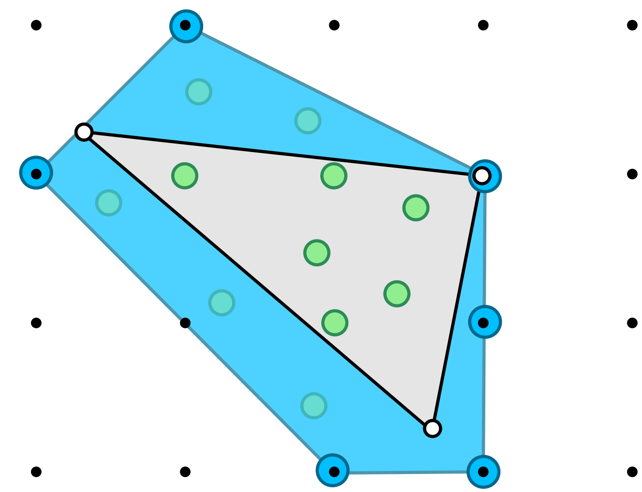

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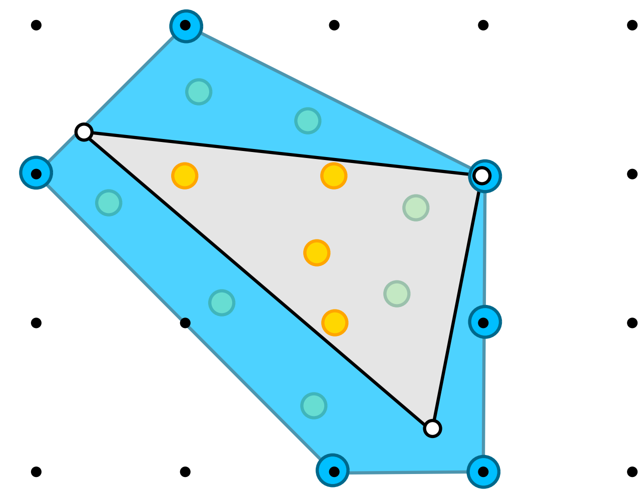

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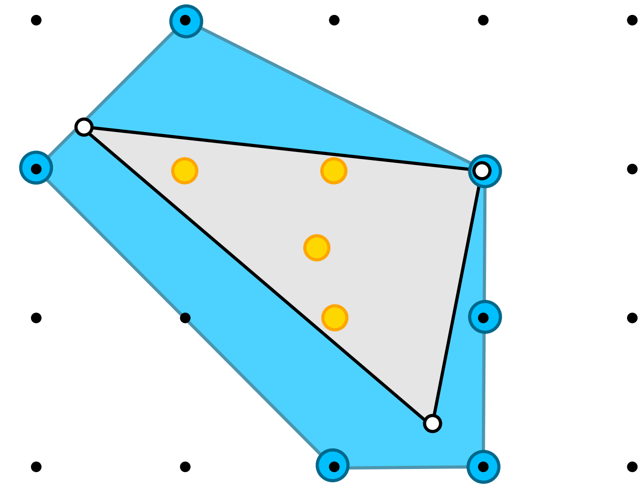
$\Rightarrow$ list of points $\mathcal{B}$ (bottom candidates)



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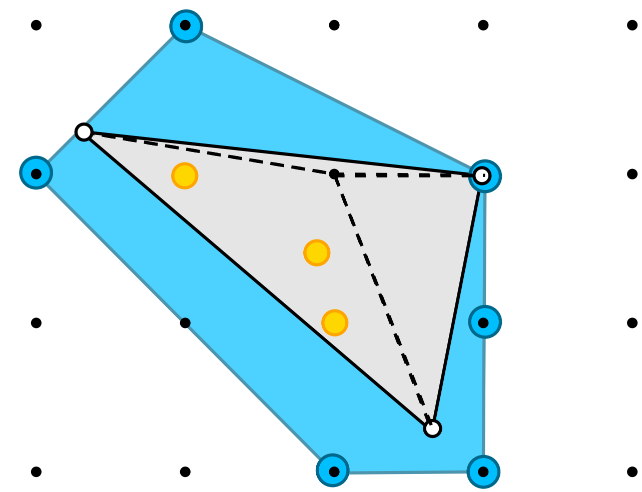

cross section at level 1

Choose a grading minimizing point from $\mathcal{B}$ and continue as before.

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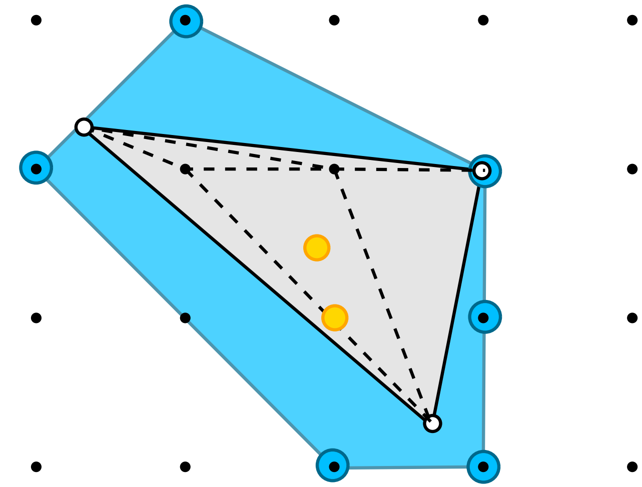

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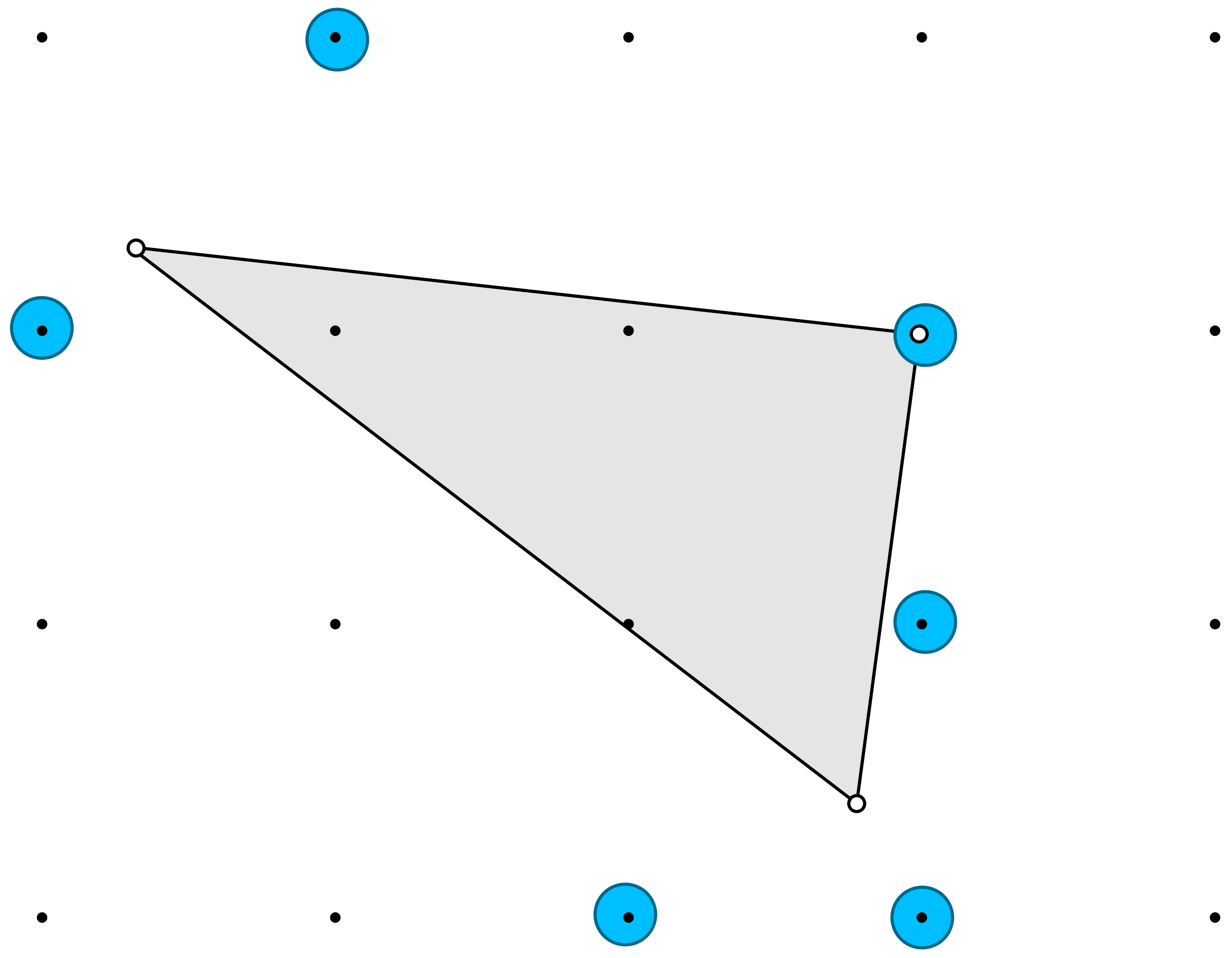
Partial Fourier-Motzkin Elimination



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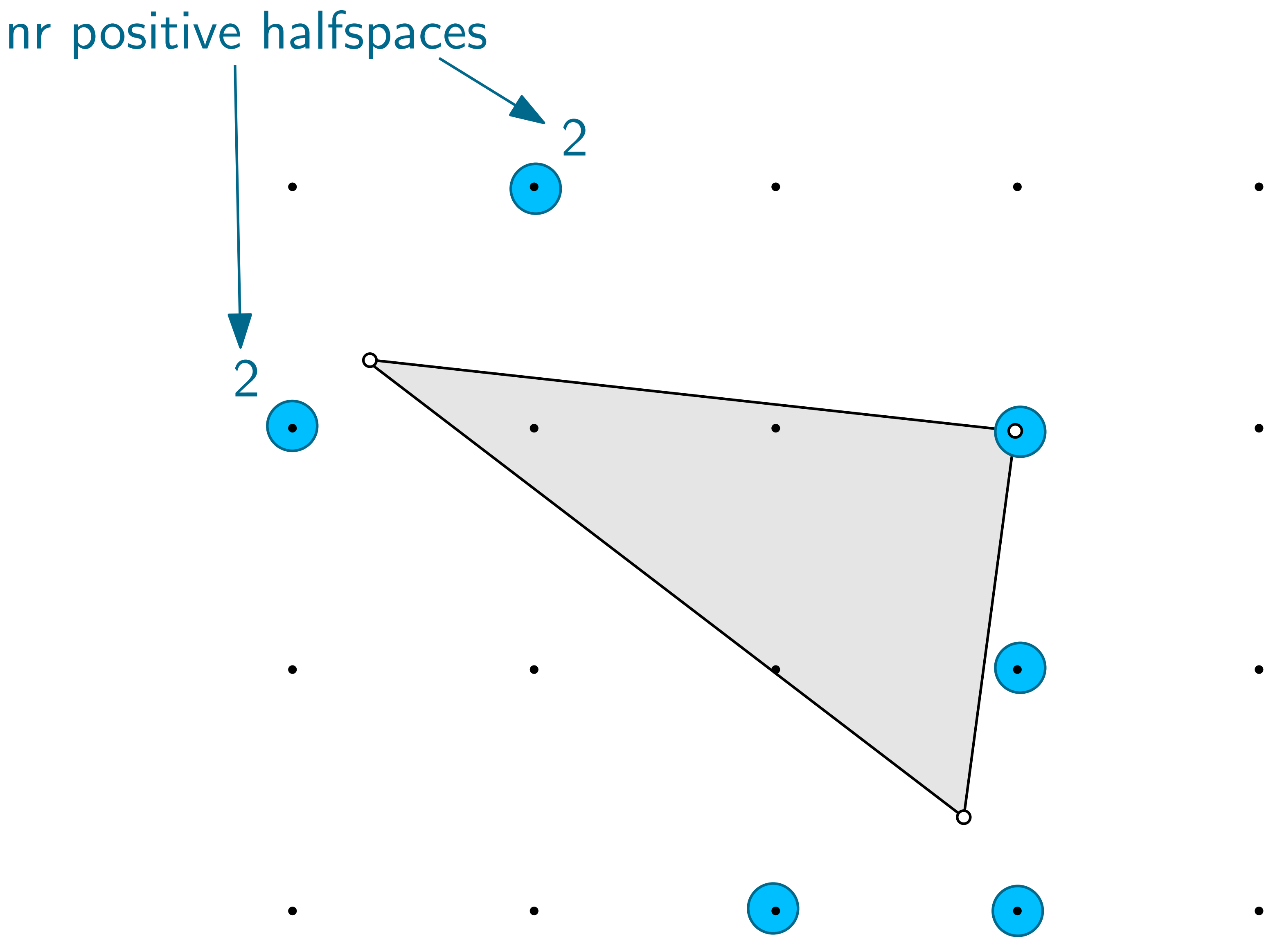

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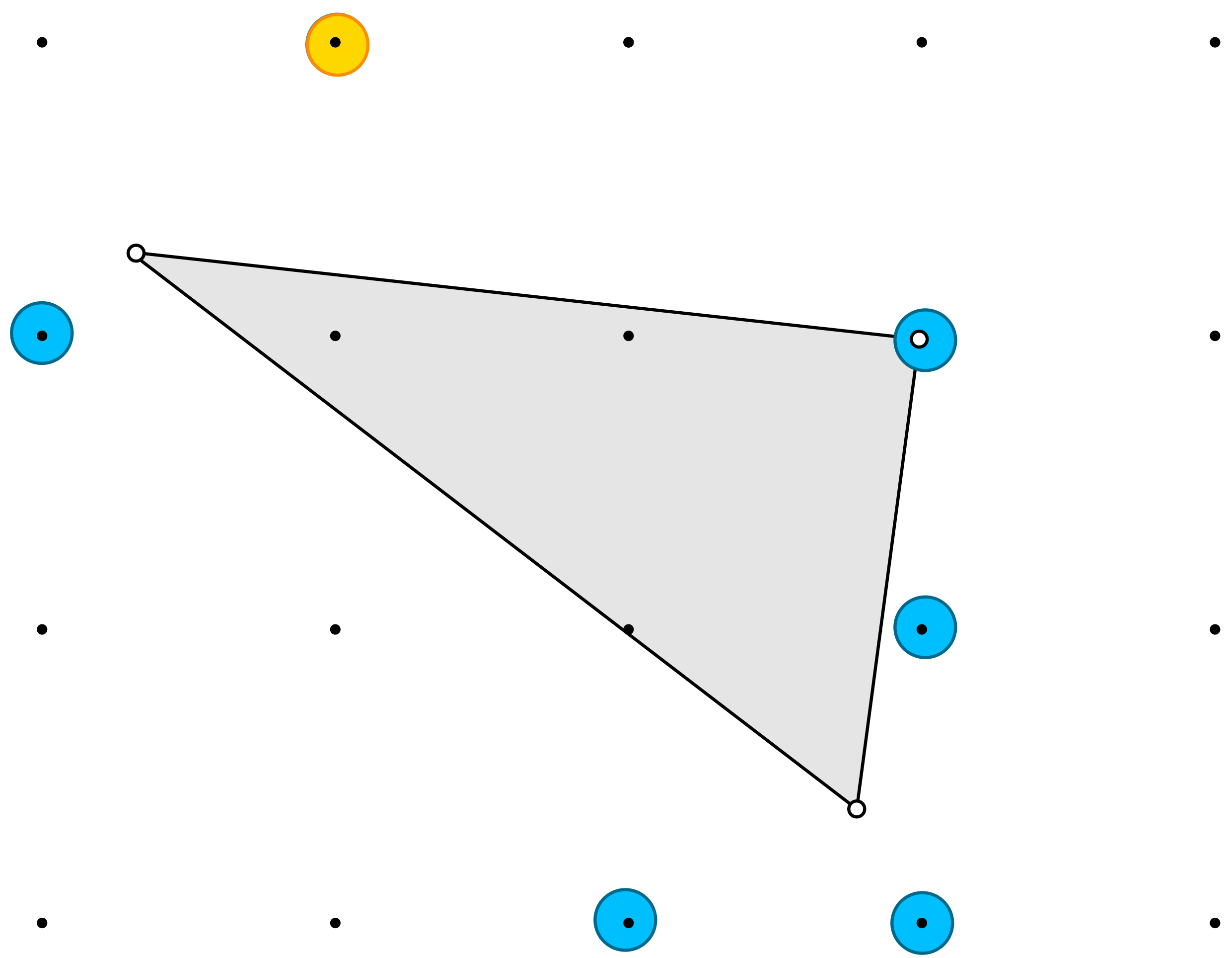

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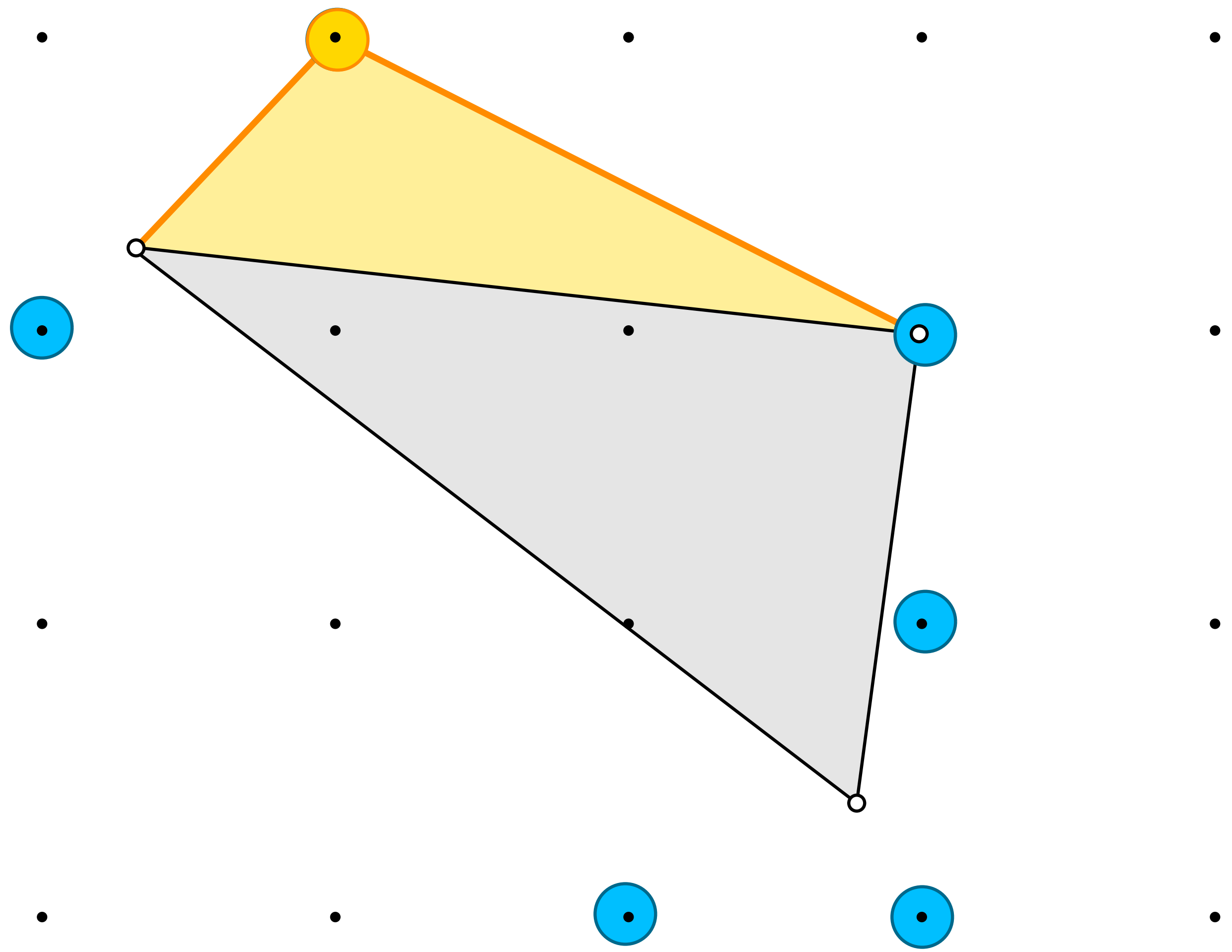

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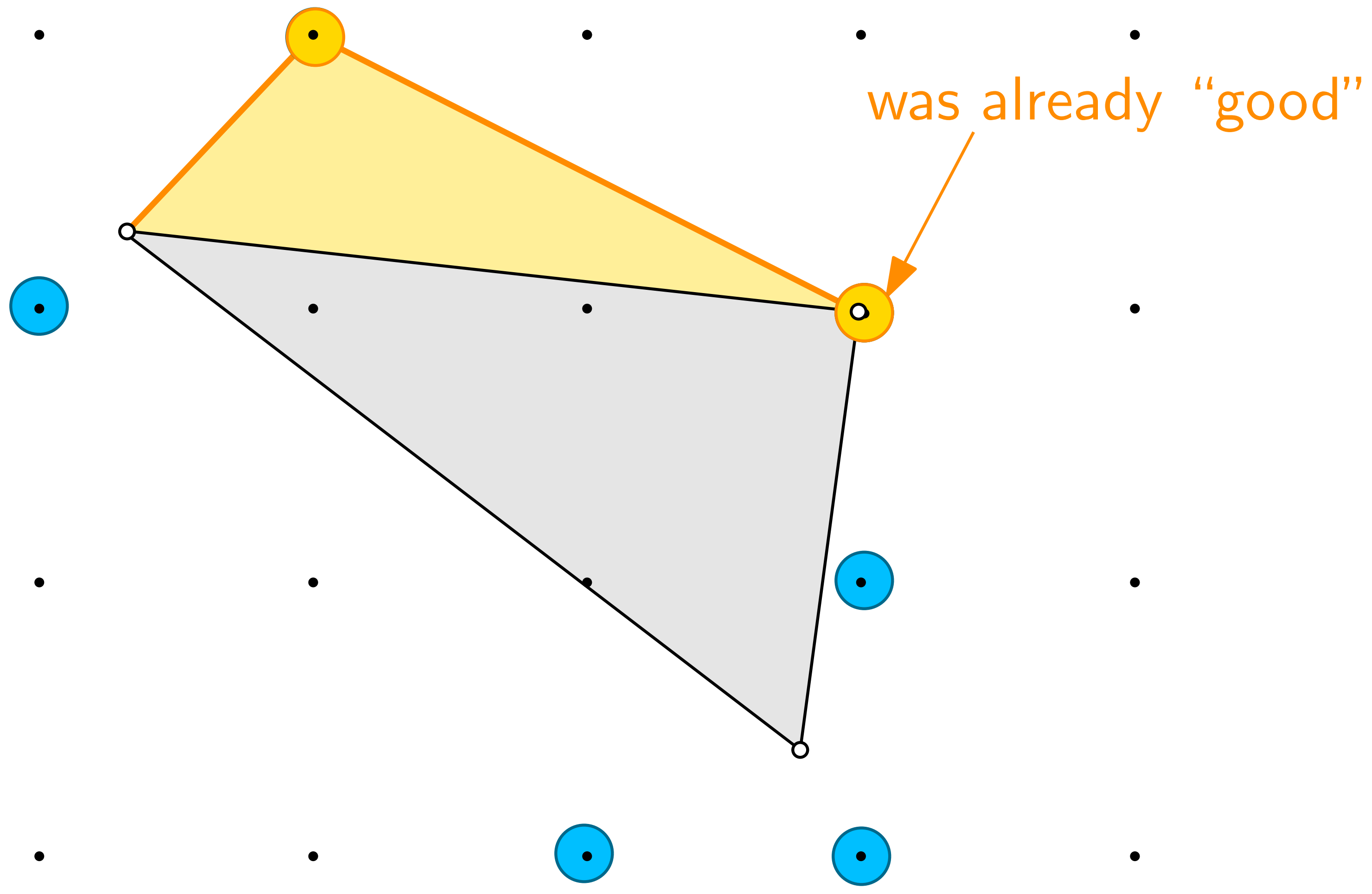

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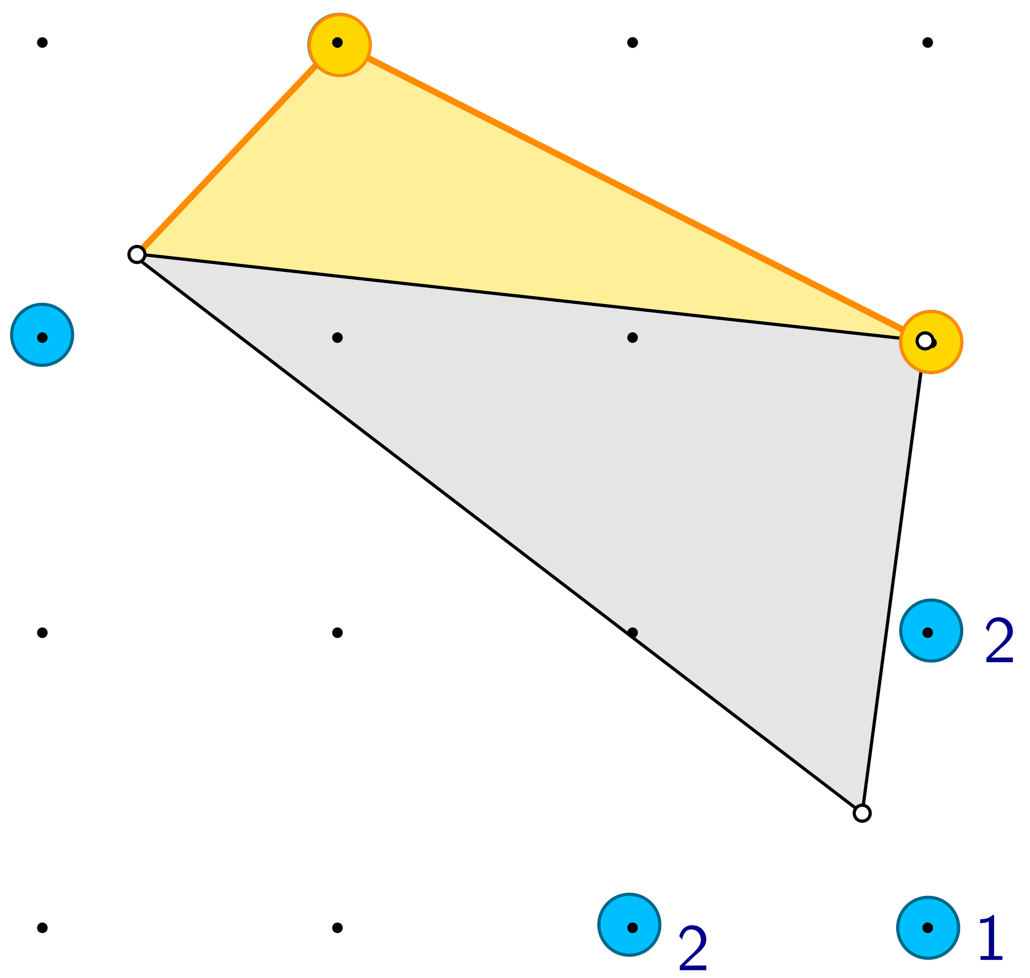

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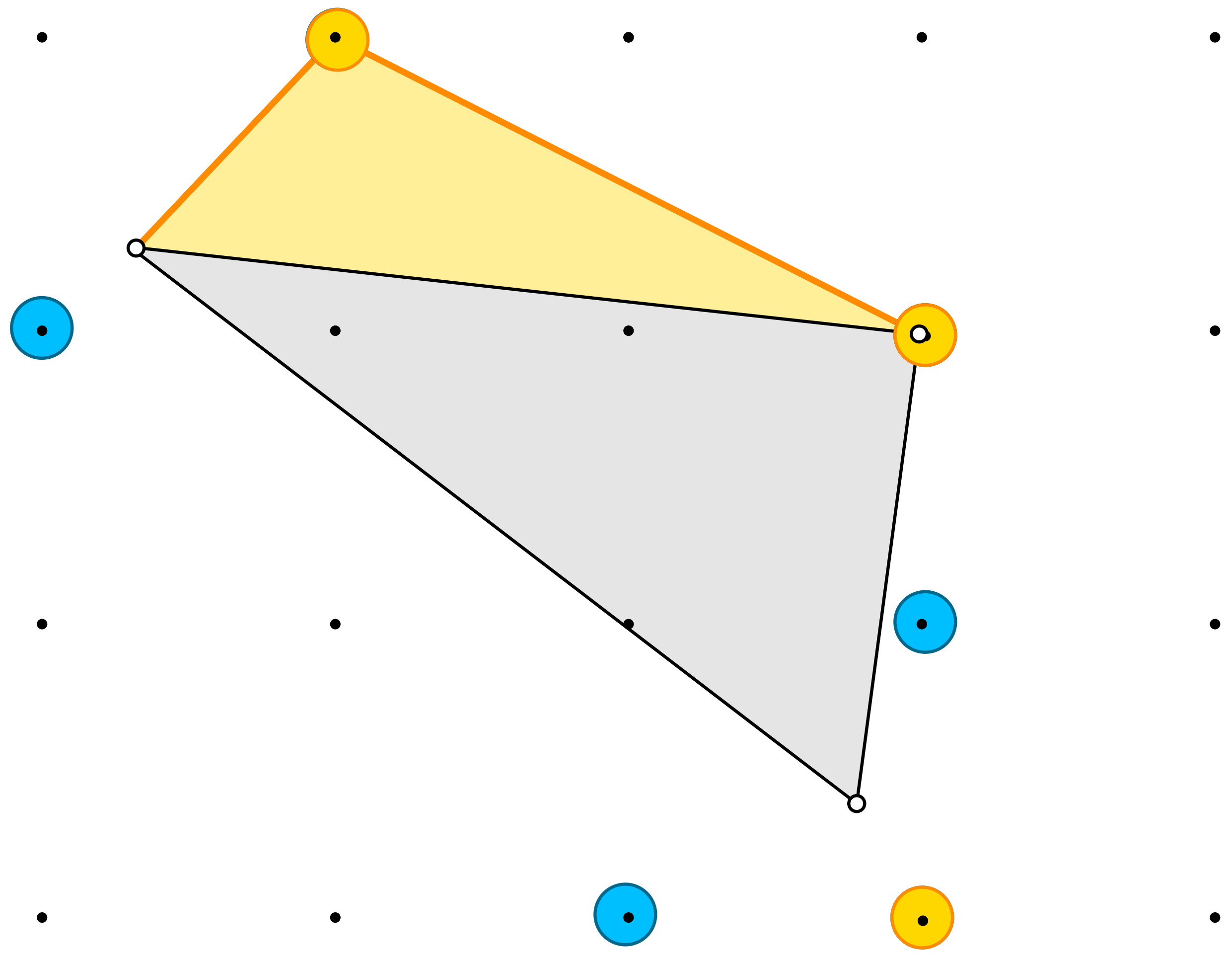

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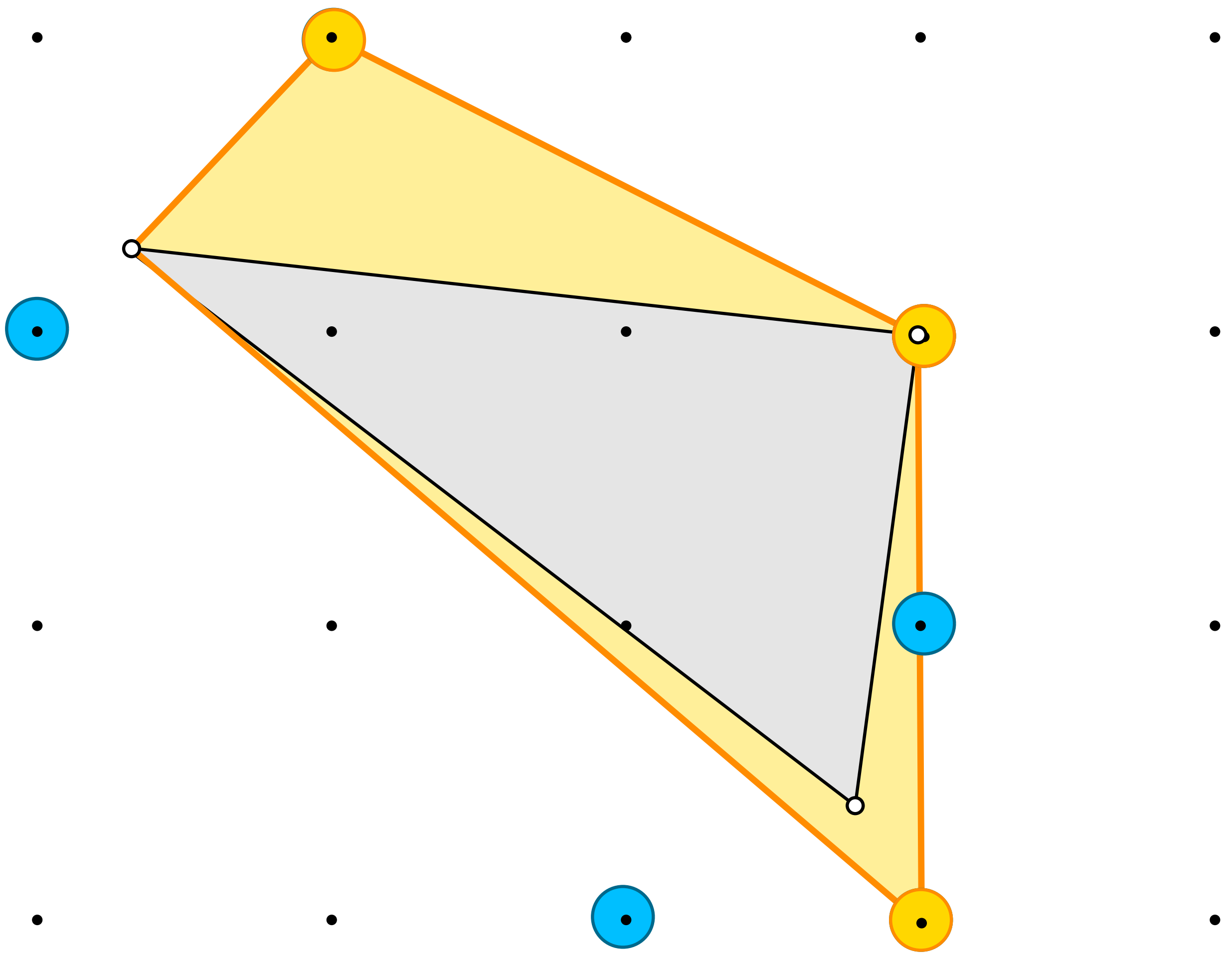

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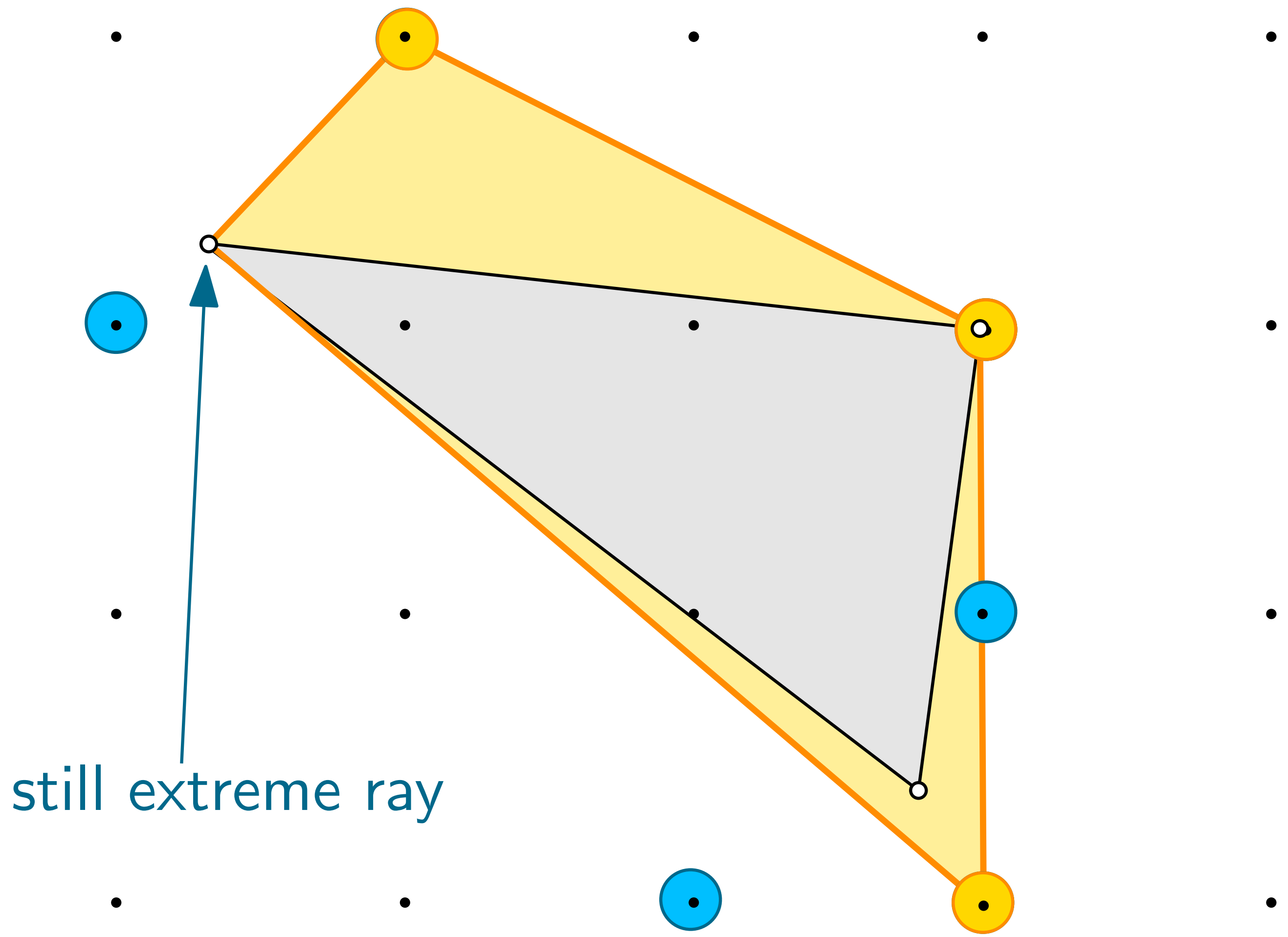

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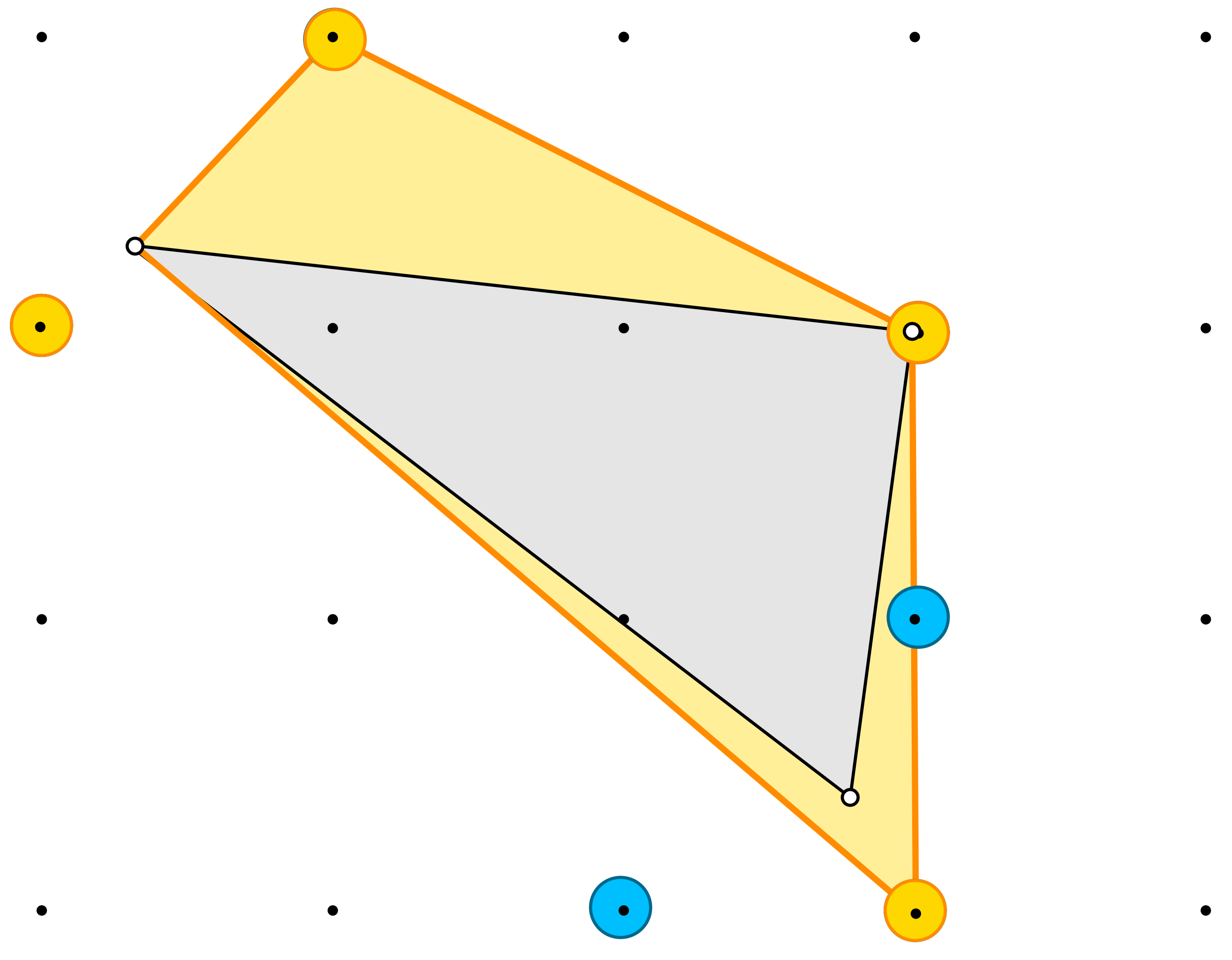

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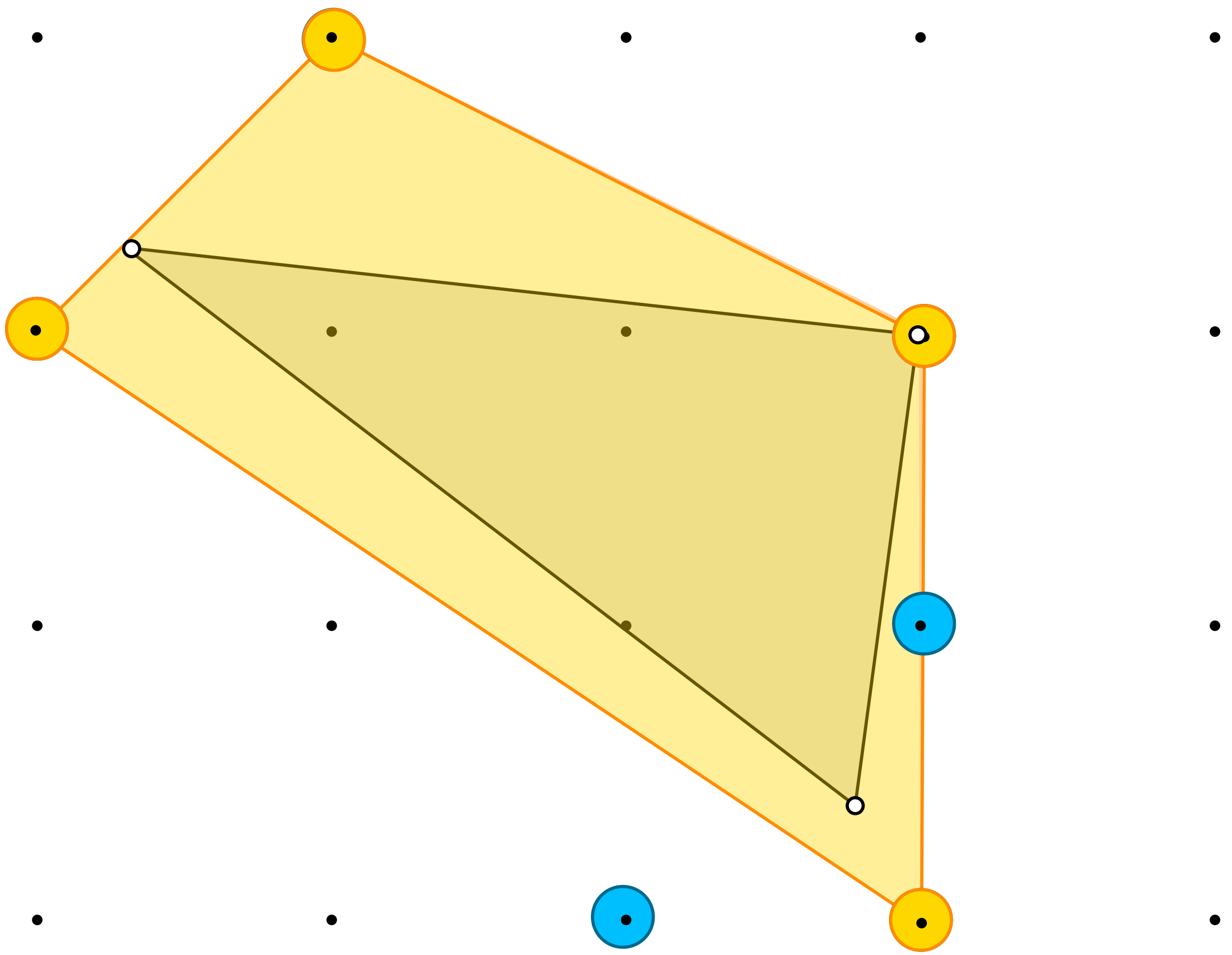

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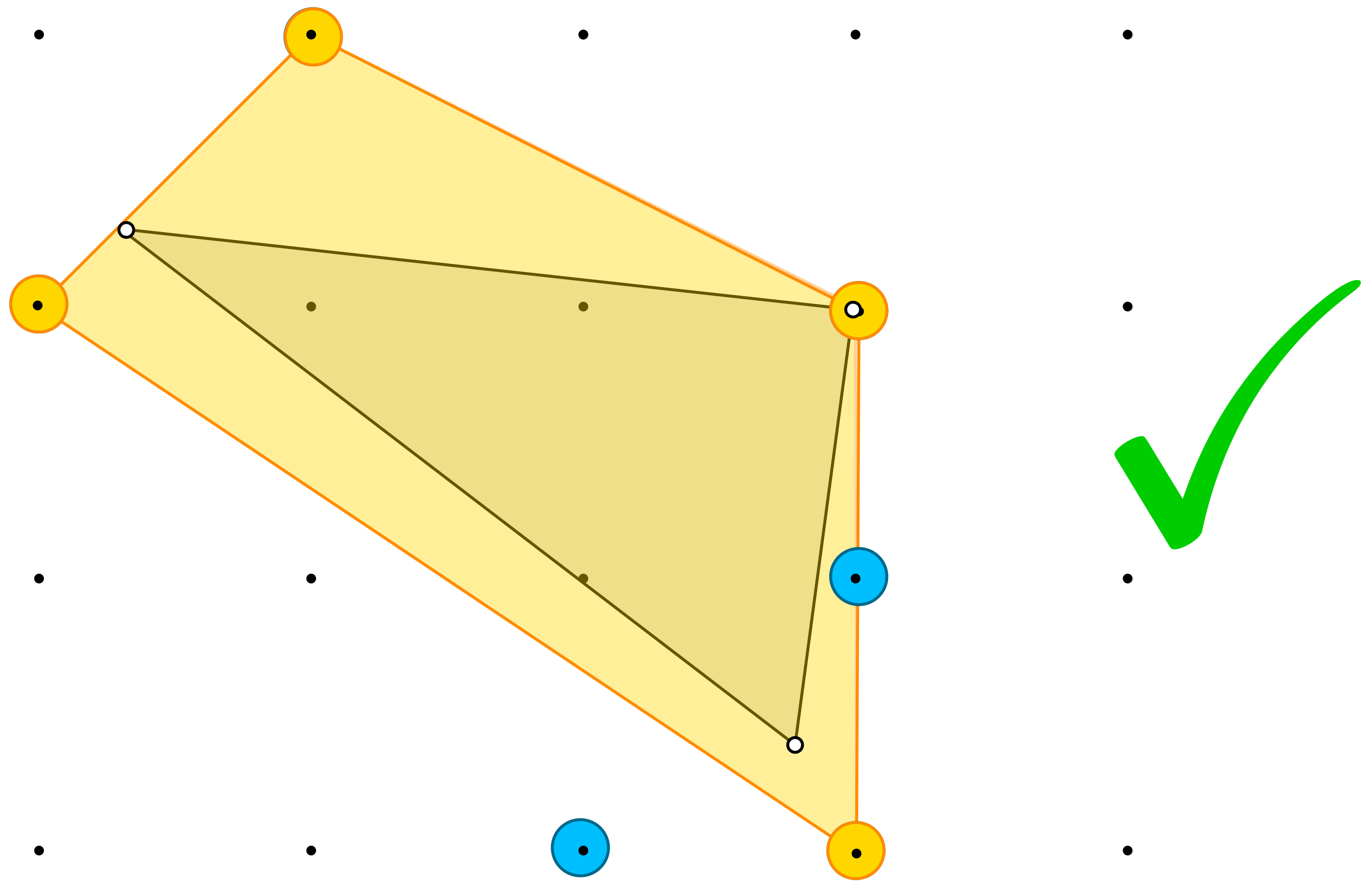

Results

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simplex vol	$9.83\,\mathrm{e}\,7$		$4.17\,\mathrm{e}\,14$		$2.8\,\mathrm{e}\,14$	
bottom vol	$8.10\,\mathrm{e}\,5$		$3.86\,\mathrm{e}\,7$		$2.02\,\mathrm{e}\,7$	
	(1)	(2)	(1)	(2)	(1)	(2)
our vol	$3.93\,\mathrm{e}\,6$	$3.93\,\mathrm{e}\,6$	$5.47\,\mathrm{e}\,7$	$8.42\,\mathrm{e}\,7$	$2.39\,\mathrm{e}\,7$	$9.36\,\mathrm{e}\,9$
factor	25	25	$7.62\,\mathrm{e}\,6$	$4.95\,\mathrm{e}\,6$	$1.09\,\mathrm{e}\,7$	$2.99\,\mathrm{e}\,4$
old time	2s		$>$12d		$>$8d	
new time	0.5s	0.4s	46s	50s	5s	2m30s

Results

	hickerson-16		hickerson-18		knapsack_11_60	
simplex vol	$9.83\,\mathrm{e}\,7$		$4.17\,\mathrm{e}\,14$		$2.8\,\mathrm{e}\,14$	
bottom vol	$8.10\,\mathrm{e}\,5$		$3.86\,\mathrm{e}\,7$		$2.02\,\mathrm{e}\,7$	
	(1)	(2)	(1)	(2)	(1)	(2)
our vol	$3.93\,\mathrm{e}\,6$	$3.93\,\mathrm{e}\,6$	$5.47\,\mathrm{e}\,7$	$8.42\,\mathrm{e}\,7$	$2.39\,\mathrm{e}\,7$	$9.36\,\mathrm{e}\,9$
factor	25	25	$7.62\,\mathrm{e}\,6$	$4.95\,\mathrm{e}\,6$	$1.09\,\mathrm{e}\,7$	$2.99\,\mathrm{e}\,4$
old time	2s		$>$12d		$>$8d	
new time	0.5s	0.4s	46s	50s	5s	2m30s

Partial Fourier-Motzkin:
no significant improvment, even in the non-simplicial case