Rigidity of toric varieties arising from graphs

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Edge ideals

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where $e = (i,j) \in E(G)$.

The kernel I_G of this morphism is called the *edge ideal*. The associated toric variety to the graph G is denoted by

$$\mathrm{TV}(G) := \mathrm{Spec}(\mathbb{C}[x_e \mid e \in E(G)]/I_G) = \mathrm{Spec}(\mathbb{C}[\sigma_G^\vee \cap M])$$

where $\sigma_G^\vee$ is called the (dual) edge cone.

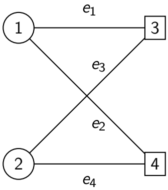
Edge ideals

Example

Let $G = K_{2,2}$ be the complete connected bipartite graph.

$$\mathbb{C}[x_1, x_2, x_3, x_4] \rightarrow \mathrm{Edr}(K_{2,2})$$

The edge ideal I_G is generated by the binomial $x_1x_4 - x_2x_3$. The three-dimensional cone $\sigma_G^\vee \subset M_\mathbb{R} \cong \mathbb{R}^4 \cap (1,1,-1,-1)^\perp$ associated to the toric variety $\mathrm{TV}(G)$ is generated by the extremal rays $[1,0,0,1]$, $[1,0,1,0]$, $[0,1,1,0]$ and $[0,1,0,1]$.



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Let $G = (V(G), A(G))$ be a finite directed graph on the vertex set $V(G) = [n] := \{1, \ldots, n\}$ with the directed edge set $A(G)$. For a directed edge $e = (i, j)$ of G, we define $\rho(e) \in \mathbb{R}^n$ by setting $\rho(e) = \mathbf{e}_i - \mathbf{e}_j$.

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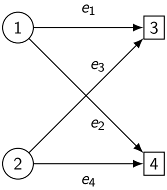
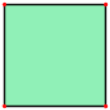
$$\mathcal{A}_G = \text{conv}\{\rho(e) : e \in A(G)\} \subset \mathbb{R}^n.$$





1. Toric varieties associated to graphs

2. Deformation Theory
   - Brief introduction
   - Deformations of toric varieties

3. Fusing three worlds

What is a deformation of an algebraic variety?

Let X be a scheme of finite type over $\mathbb{C}$ and let A be an Artinian algebra over $\mathbb{C}$. An infinitesimal deformation of X over A is defined as the following cartesian diagram:

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(\mathbb{C}) & \hookrightarrow & \mathrm{Spec}(A) \end{array}$$

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where π is flat. Let $\pi' : X \to \mathcal{X}'$ be another deformation of X over $\mathrm{Spec}(A)$. We say that π and π' are isomorphic, if there exists a map $\mathcal{X} \to \mathcal{X}'$ over $\mathrm{Spec}(A)$ inducing the identity on X.

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Let Def_X be a functor such that $Def_X(A)$ is the set of deformations of X over $\mathrm{Spec}(A)$ modulo isomorphisms.

What is rigidity?

Definition

The map π is called a first order deformation of X_0 if $S = \mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$.
We set $T^1_{X_0} := \mathrm{Def}_{X_0}(\mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2)))$.

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i.e. every deformation $\pi \in \mathrm{Def}_{X_0}(A)$ over a Artin ring A is isomorphic to the trivial deformation $X_0 \times \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A)$.

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Setup for the first order deformations

This section serves as a sketch for the proofs on deformations of toric varieties arising from graphs.

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- Fix a deformation degree $R \in M$.
- Let $\sigma = \operatorname{Cone}(a_1, \ldots, a_n) \subseteq N_{\mathbb{R}}$.

Consider the following affine space

$$[R = 1] := \{a \in N_{\mathbb{Q}} \mid \langle R, a \rangle = 1\} \subseteq N_{\mathbb{Q}}.$$

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$$[R = 1] := \{a \in N_{\mathbb{Q}} \mid \langle R, a \rangle = 1\} \subseteq N_{\mathbb{Q}}.$$

- The *crosscut* of σ in degree R: $Q(R) := \sigma \cap [R = 1]$ in the assigned vector space $[R = 0]$.

Setup for the first order deformations

• For each two-dimensional compact face of $Q(R)$ assign sign vectors ϵ_i on the compact edges d^i such that the oriented edges form a cycle

$$\sum_{i\in[N]} \overline{\epsilon_i} d^i = 0$$

.

• Define the related vector space to a deformation degree $R \in M$ as

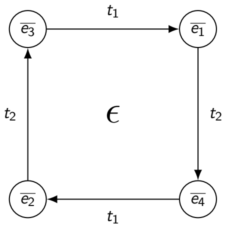
$$V(R) = \{\overline{t} = (t_1, \ldots, t_N) \in \mathbb{C}^N | \sum_{i\in[N]} t_i \overline{\epsilon_i} d^i = 0, \forall \epsilon \preceq Q(R)\}.$$

Setup for the first order deformations

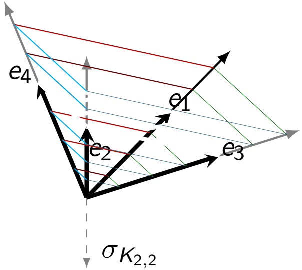
Example (continued)

Let us consider the dual edge cone $\sigma^{\vee}_{K_{2,2}} = \mathrm{Cone}([1,0,0,1],[1,0,1,0],[0,1,1,0],[0,1,0,1]) \subset M_{\mathbb{R}}$ from the previous example. We calculate $\sigma_G = \mathrm{Cone}(e_1, e_2, e_3, e_4) \subset N_{\mathbb{R}}$. For the deformation degree $R = [1,1,1,1] \in M$, we obtain the compact crosscut $Q(R) = \mathrm{conv}(e_1, e_2, e_3, e_4)$.

Setup for the first order deformations



$K_{2,2}$



$$V(R) = \{(t_1, t_2, t_1, t_2) \in \mathbb{C}^4 | \sum_{i \in [4]} t_i \overline{\epsilon_i} d^i = 0\}.$$

Formula for $T^1_{X_0}$

Theorem (Corollary 2.7, [Alt00])

If the affine normal toric variety X_0 is smooth in codimension 2, then $T^1_{X_0}(-R)$ is contained in $V(R)/\mathbb{C}(1,\ldots,1)$. Moreover, it is built by those vectors $\overline{t}$ satisfying $t_{ij} = t_{jk}$ where $\overline{a_j}$ is a non-lattice common vertex in $Q(R)$ of the edges $d^{ij} = \overline{\overline{a_i}\ \overline{a_j}}$ and $d^{jk} = \overline{\overline{a_j}\ \overline{a_k}}$.

Formula for $T^1_{X_0}$

Example (continued)

We obtain that $T^1_{\mathrm{TV}(K_{2,2})}([1,1,1,1]) = \{(t_1, t_2, t_1, t_2) \in \mathbb{C}^4\}/\mathbb{C}(1,1,1,1)$ and $T^1_{\mathrm{TV}(K_{2,2})} \neq 0$. Hence the toric variety $\mathrm{TV}(K_{2,2})$ is not rigid.

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Remark

For the application of this theorem, we need an explicit description of the faces of the edge cone $\sigma_G \subset N_{\mathbb{R}}$ (in particular extremal rays, two and three faces). It is not straightforward from the construction of $\mathrm{TV}(G)$ as in the case for the dual edge cone $\sigma_G^\vee \subset M_{\mathbb{R}}$.

Faces of the edge cone σ_G via graphs

Any proper face of σ_G can be characterized via subgraphs arising from so-called *first independent sets* of G. Let us first consider the 1-faces (extremal rays) of σ_G.

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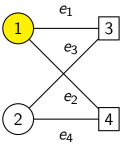
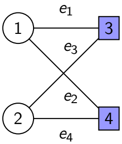
- A nonempty subset A of $V(G)$ is called an *independent set* if it contains no adjacent vertices.



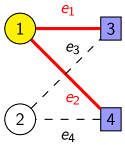
- The *neighbor set* of $A \subseteq V(G)$ is defined as

$$N(A) := \{v \in V(G) \mid v \text{ is adjacent to some vertex in } A\}.$$



First independent sets of G and the extremal rays of σ_G

- The *associated subgraph* G$\{A\}$ is defined as the spanning induced subgraph of $A \cup N(A)$.
  Example : $G\{A\} = G[A \cup N(A)] \cup \{2\}$.



The first independent sets of G are

$\mathcal{I}_G^{(1)} :=$ {Two-sided maximal independent sets and one-sided independent sets $U_i \backslash \{\bullet\}$ where their associated graph has two connected components}

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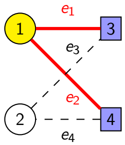

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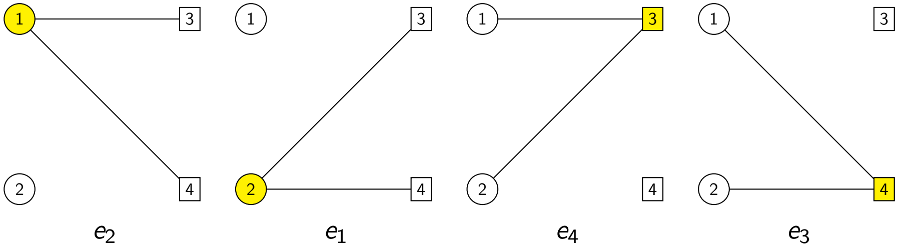
Theorem ([Por21], Theorem 2.8)

$$\begin{aligned} \pi : \mathcal{I}_G^{(1)} &\longrightarrow \sigma_G^{(1)} \\ A &\mapsto \mathfrak{a} := (\mathcal{H}_{A_i} \cap \sigma_G^{\vee})^* \end{aligned}$$

for a fixed $i \in \{1, 2\}$ *with* $A_i \neq \emptyset$.

First independent sets of G and the extremal rays of σ_G

Example (continued)



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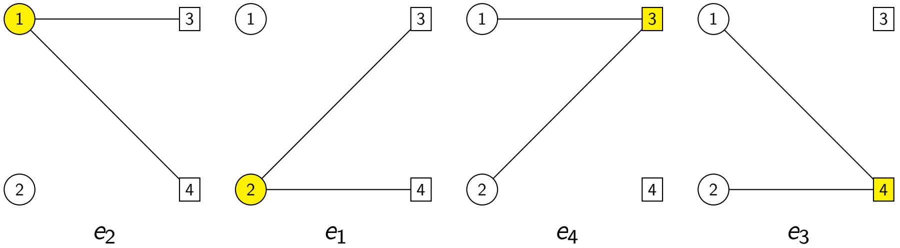

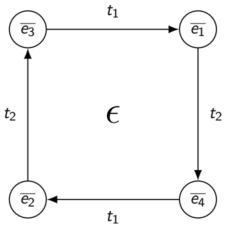

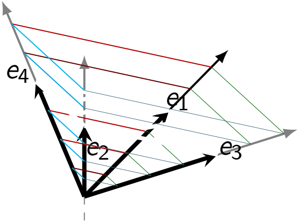

Faces of σ_G via first independent sets

Theorem ([Por21], Theorem 2.18)

Let $S \subseteq \mathcal{I}_G^{(1)}$ be a subset of d first independent sets and let π be the bijection from the previous theorem. The extremal ray generators $\pi(S)$ span a face of dimension d if and only if $\mathsf{G}[S] := \bigcap_{A \in S} \mathsf{G}\{A\}$ has $d+1$ connected components.

Faces of σ_G via first independent sets

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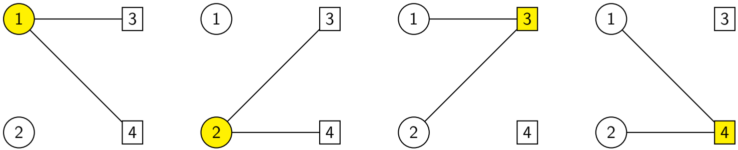
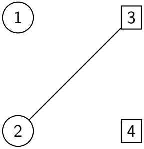
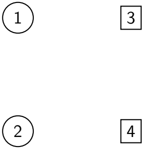


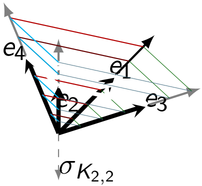
e_2 e_1 e_4 e_3



e_1 and e_4 span a 2-face



e_1 and e_2 do not span a 2-face



$\sigma_{K_{2,2}}$

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   - Bipartite graph G and the faces of σ_G
   - Bipartite graph G and the deformations of $\mathrm{TV}(G)$
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Some criteria for rigidity in terms of graphs

Some applications of the face theorem and combinatorial T^1 formula:

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Theorem

Let $G \subseteq K_{m,n}$ be a connected bipartite graph and let $\tau \preceq \sigma_G$ be a three-dimensional non-simplicial face of the edge cone σ_G. Then τ is spanned by four extremal rays and $\mathrm{TV}(G)$ is not rigid.

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Example (continued)

The edge cone $\sigma_{K_{2,2}}$ has a non-simplicial three dimensional face and hence $\mathrm{TV}(K_{2,2})$ is not rigid.

Some criteria for rigidity in terms of graphs

Theorem

Let $G \subsetneq K_{m,n}$ be a connected bipartite graph with exactly one two-sided first independent set $A = A_1 \cup A_2 \in \mathcal{I}_G^{(1)}$ with $A_1 \subset U_1$ and $A_2 \subset U_2$. Then

1. $\mathrm{TV}(G)$ *is not rigid, if* $|A_1| = 1$ *and* $|A_2| = n - 2$ *or*
   if $|A_1| = m - 2$ *and* $|A_2| = 1$.
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Example

Let $G \subset K_{4,4}$ be the connected bipartite graph obtained by one edge removal from the complete bipartite graph $K_{4,4}$. Then $\mathrm{TV}(G)$ is rigid.

- This is a generalization of the result by Bigdeli, Herzog, Lu in [BHL15].
- *Check* `https://github.com/iremportakal` *for a function which receives the dual edge cone and outputs the information about rigidity of the associated toric variety.*

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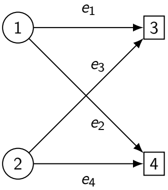

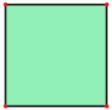

Fano polytopes

Let $\mathcal{P} \subset \mathbb{R}^n$ be a full-dimensional lattice polytope.

- We say that $\mathcal{P}$ is a *Fano* if the origin of $\mathbb{R}^n$ belongs to the interior of $\mathcal{P}$ and the vertices of $\mathcal{P}$ are primitive lattice points in $\mathbb{Z}^n$.
- A Fano polytope $\mathcal{P}$ is called *terminal* if every lattice point on the boundary is a vertex.
- A Fano polytope $\mathcal{P}$ is said to be *reflexive* if each facet of $\mathcal{P}$ has lattice distance one from the origin. Equivalently, its dual polytope

$$\mathcal{P}^\vee = \{x \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{y} \rangle \geqslant -1 \text{ for all } y \in \mathcal{P}\}$$

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  is a lattice polytope.
- A Fano polytope is called $\mathbb{Q}$-*factorial* if it is simplicial.
- A Fano polytope is called *smooth* if the vertices of each facet form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Edge polytopes

Proposition ([HHMNO11, Proposition 3.2])

Let G be a finite directed graph. Then the following arguments are equivalent:

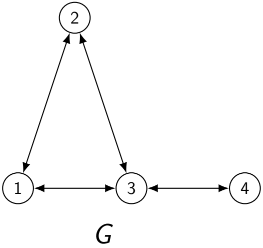
1. *$\mathcal{A}_G$ is Fano;*
2. *$\mathcal{A}_G$ is terminal reflexive;*
3. *Every directed edge of G belongs to a directed cycle in G.*

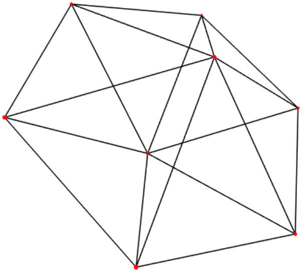
In this case, X_G is a Gorenstein toric Fano variety with terminal singularities.

Notation: For a Fano polytope $\mathcal{A}_G$, denote X_G the normal toric variety associated to the spanning fan of $\mathcal{A}_G$.

Symmetric edge polytope

Example





Theorem ([Hig15, Theorem 2.2 and Corollary 2.3])

Let G be a finite symmetric directed graph on $[n]$. Then the following arguments are equivalent:

1. X_G *is smooth;*
2. X_G *is $\mathbb{Q}$-factorial;*
3. G^{un} *has no even cycle as subgraphs.*

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1. *X_G is smooth;*
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(Bien and Brion ,1996) proved that every smooth toric Fano variety is rigid. (de Fernex and Hacon, 2011) proved the rigidity of $\mathbb{Q}$-factorial terminal toric Fano varieties.

Rigidity for Gorenstein toric Fano varieties

The most general rigidity theorem for toric Fano varieties known to this date is the following result of Totaro:

Theorem ([Tot12, Theorem 5.1])

A toric Fano variety which is smooth in codimension 2 and $\mathbb{Q}$-factorial in codimension 3 is rigid.

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If $\mathcal{P}$ is Fano polytope, then $X_{\mathcal{P}}$ satisfies this condition if and only if all 2-faces of $\mathcal{P}$ are triangles and each edge of $\mathcal{P}$ has lattice length 1 and is contained in some hyperplane which has height 1 with respect to the origin, then $X_{\mathcal{P}}$.

Lemma (Kara, P., Tsuchiya, 2021)

Let G be a symmetric directed graph. If the underlying graph of G has no C_4 (cycle graph with 4 vertices) as a subgraph, then $\mathcal{A}_G$ has all triangle 2-faces and hence X_G is rigid.

Lemma (Kara, P., Tsuchiya, 2021)

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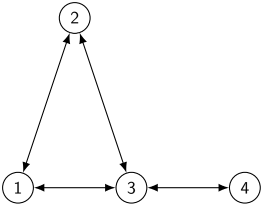
Example (continued)

The Gorenstein toric Fano variety X_G is rigid, since G has a no C_4 as a subgraph equivalently $\mathcal{A}$ has a square 2-face.



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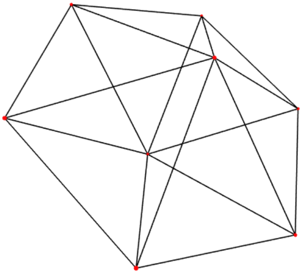

Thank you for your time!

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