Defect Polytopes and Dual Defective Toric Varieties

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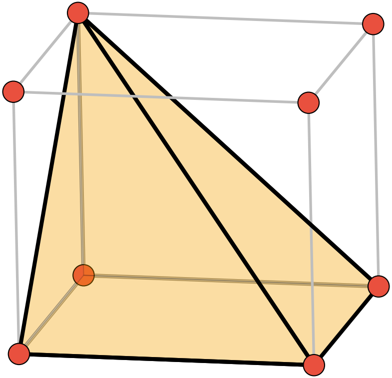

Michael Joswig and

Andreas Paffenholz

TU Darmstadt, Germany

Dual Defective Varieties

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- X projective variety of dimension n
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- X dual defective $:\Longleftrightarrow m - \dim X^\vee > 0$

Toric Varieties

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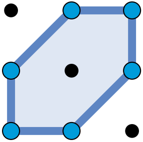

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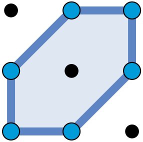

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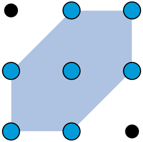

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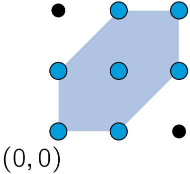

- X_A projective toric variety defined by A :

  closure of $(\mathbb{C}^n)^* \longrightarrow \mathbb{P}^{m-1}$

  $$t \longmapsto [\, t^{a_1} : \dots : t^{a_m} \,]$$

$$(t_1, t_2) \longmapsto [1 : t_1 : t_2 : t_1 t_2 : t_1^2 t_2 : t_1 t_2^2 : t_1^2 t_2^2]$$

Defect Polytopes

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- P lattice polytope, $A := P \cap \mathbb{Z}^n = \{a_1, \dots, a_m\}$
- X_A associated toric variety in $\mathbb{P}^{m-1}$

- A-discriminant: $\Delta_A \in \mathbb{C}[x_1, \dots, x_m]$ irreducible such that
  - Δ_A is the defining polynomial if $(X_A)^\vee$ is a hypersurface, and
  - $\Delta_A := 1$ otherwise.

[Gelfand, Kapranov, Zelevinsky]

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- P defect polytope $:\iff X_A$ dual defective

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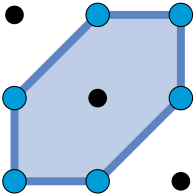

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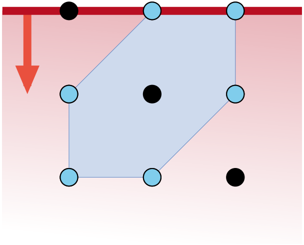

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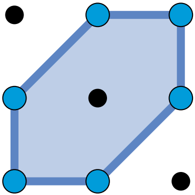

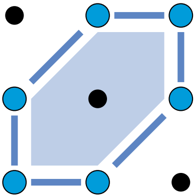

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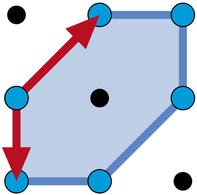
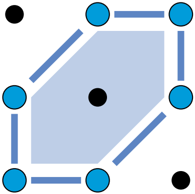


- P smooth $:\Longleftrightarrow$ X_A non-singular
  $:\Longleftrightarrow$ primitive edge directions at any vertex span $\mathbb{Z}^n$



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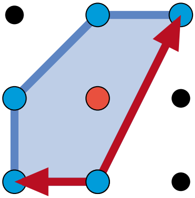

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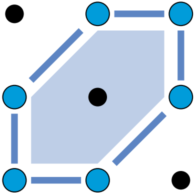

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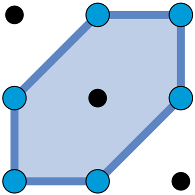

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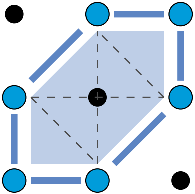

- lattice volume $\operatorname{lvol}(F)$ of a face F of P
  - normalized volume w.r.t. $\mathbb{Z}^F := \operatorname{aff} F \cap \mathbb{Z}^n$
  - $\operatorname{lvol}(\text{unit simplex}) = 1$ in any dimension

Combinatorial Dual Defect

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- $P \subseteq \mathbb{R}^n$ lattice polytope
- $\mathrm{F}_P(k)$ k-dimensional faces of P

$$c(P) := \sum_{k=0}^{n} (-1)^{n-k} (k+1) \sum_{F \in \mathrm{F}_P(k)} \mathrm{lvol}(F)$$

[Gelfand, Kapranov, Zelevinsky]

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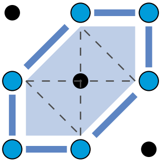

$$c(P) \;=\; 3{\cdot}6 + 6{\cdot}(-2{\cdot}1) + 6{\cdot}1 \;=\; 12$$

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- **Theorem:** [Di Rocco]
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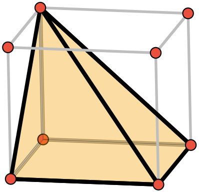
- Proposition : $c_r(\mathrm{pyr}^r([0,1]^3)) = -r! < 0$



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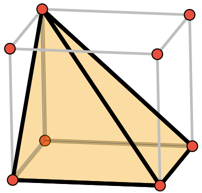

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  - computer search with **polymake**
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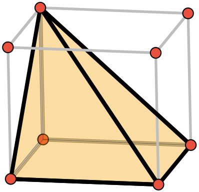
    - fully programmable interface



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$$f(P,t) := (-1)^n \sum_{k=0}^{n} (-1)^k (k+1)! \sum_{F \in \mathrm{F}_k(P)} \underbrace{|t \cdot F \cap \mathbb{Z}^F|}_{\mathrm{ehr}(F,t)} t^{n-k}$$

[Di Rocco]

- **Conjecture**: Coefficients of $f(P,t)$ are non-negative
- **Proposition**: leading coefficient of $f(\mathrm{pyr}^r([0,1]^3),t) < 0$ for $r \geq 1$.

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[Di Rocco]

- Question : $c_1(P) = 0 \implies$ strict Cayley for all lattice polytopes?
- Proposition : $c_1(\mathrm{pyr}^2([0,1]^2)) = 0$, but not strict Cayley

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Dual Defect and Spectral Values

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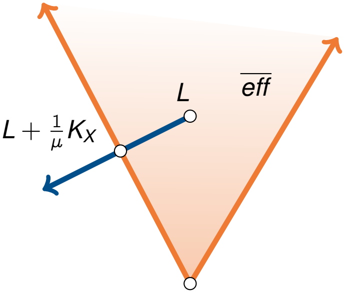
- X_A projectively embedded toric variety



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- adjoint line bundles $L + c \cdot K_X$ for $c \geq 0$

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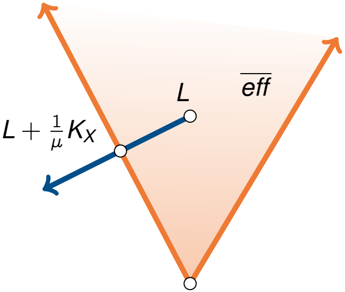

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- spectral value: $\mu := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ big}))^{-1}$

$$= (\max(c \in \mathbb{R} \mid Mx \le b - c\mathbf{1} \text{ has a solution}))^{-1}$$

Dual Defect and Spectral Values

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- Theorem: If $\mu \geq \frac{3n+4}{4}$ then X_A is dual defective

  [Di Rocco, Haase, Nill, P.]

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  - X_A has dual defect $r - m$ [Dickenstein, Feichtner, Sturmfels]

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  $\Longrightarrow$ vertices of P are on two parallel hyperplanes

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- P Cayley polytope of order m: (lattice projection to unit $(m-1)$-simplex)

  $\Longrightarrow$ vertices of P are on two parallel hyperplanes

- If X_A dual defective then vertices of P are on two parallel hyperplanes

  [Curran, Cattani], [Esterov]

Dual Defect and Spectral Values

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- Theorem: If $\mu \geq \frac{3n+4}{4}$ then X_A is dual defective

  [Di Rocco, Haase, Nill, P.]

  - P Cayley polytope of order m in dimension $r > m$
  - X_A has dual defect $r - m$ [Dickenstein, Feichtner, Sturmfels]

- P Cayley polytope of order m: (lattice projection to unit $(m-1)$-simplex)

  $\Longrightarrow$ vertices of P are on two parallel hyperplanes

- If X_A dual defective then vertices of P are on two parallel hyperplanes

  [Curran, Cattani], [Esterov]

- Question Does $\mu > \frac{n+2}{2}$ suffice?

polymake

polymake: Software for research in discrete geometry

[Assarf, Gawrilow, Herr, Herrmann, Joswig, Lorenz, Möser, P., Rehn, Rörig, Schröter]

- polytopes, point configurations, lattice polytopes and toric geometry
- simplicial complexes, tropical geometry, ...

- **http://polymake.org**
  - using interfaces to **normaliz2** [Bruns et al.], **latte** [Köppe et al.]

- methods used here:
  - extension DefectPolytopes
    http://polymake.org/polytopes/paffenholz/data/polymake/extensions/DefectPolytopes
  - extension PolyhedralAdjunction
    http://polymake.org/polytopes/paffenholz/data/polymake/extensions/PolyhedralAdjunction