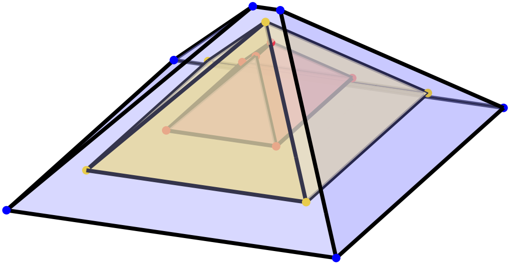
Polyhedral Adjunction Theory

Hannover, February 2012

TECHNISCHE UNIVERSITÄT DARMSTADT



Andreas Paffenholz

joint with

Sandra Di Rocco (Stockholm)

Christian Haase (Frankfurt)

Benjamin Nill (Case Western)

arXiv:1105.2415

Classical Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Polarized variety: pair (X, L), where
  - X is a projective variety of dimension n
  - L is an ample line bundle on X.
  - assume that X is $\mathbb{Q}$-Gorenstein, i.e. K_X is $\mathbb{Q}$-Cartier

Classical Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Polarized variety: pair (X, L), where
  - X is a projective variety of dimension n
  - L is an ample line bundle on X.
  - assume that X is $\mathbb{Q}$-Gorenstein, i.e. K_X is $\mathbb{Q}$-Cartier
- adjunction theory: study adjoint line bundles $t \cdot L + \quad K_X$

Classical Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Polarized variety: pair (X, L), where
  - X is a projective variety of dimension n
  - L is an ample line bundle on X.
  - assume that X is $\mathbb{Q}$-Gorenstein, i.e. K_X is $\mathbb{Q}$-Cartier
- adjunction theory: study adjoint line bundles $L + {}^{1}/_{t} \cdot K_X$

Classical Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Polarized variety: pair (X, L), where
  - X is a projective variety of dimension n
  - L is an ample line bundle on X.
  - assume that X is $\mathbb{Q}$-Gorenstein, i.e. K_X is $\mathbb{Q}$-Cartier
- adjunction theory: study adjoint line bundles $\quad L + {}^{1}/_{t} \cdot K_X$
- two invariants
  - nef-value: $\quad \tau := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ ample }))^{-1}$

Classical Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Polarized variety: pair (X, L), where
  - X is a projective variety of dimension n
  - L is an ample line bundle on X.
  - assume that X is $\mathbb{Q}$-Gorenstein, i.e. K_X is $\mathbb{Q}$-Cartier
- adjunction theory: study adjoint line bundles $L + {}^1\!/t \cdot K_X$
- two invariants
  - nef-value: $\tau := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ ample }))^{-1}$
  - unnormalized spectral value: $\mu := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ big }))^{-1}$

Classical Adjunction Theory

- Polarized variety: pair (X, L), where
  - X is a projective variety of dimension n
  - L is an ample line bundle on X.
  - assume that X is $\mathbb{Q}$-Gorenstein, i.e. K_X is $\mathbb{Q}$-Cartier
- adjunction theory: study adjoint line bundles $\quad L + {}^{1}/_{t} \cdot K_X$
- two invariants
  - nef-value: $\quad \tau := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ ample }))^{-1}$
  - unnormalized spectral value: $\quad \mu := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ big }))^{-1}$

$$||$$

$$(\text{ effective threshold })^{-1}$$

Classical Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Polarized variety: pair (X, L), where
  - X is a projective variety of dimension n
  - L is an ample line bundle on X.
  - assume that X is $\mathbb{Q}$-Gorenstein, i.e. K_X is $\mathbb{Q}$-Cartier
- adjunction theory: study adjoint line bundles $L + {}^{1}/_{t} \cdot K_X$
- two invariants
  - nef-value: $\tau := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ ample }))^{-1}$
  - unnormalized spectral value: $\mu := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ big }))^{-1}$

    $||$

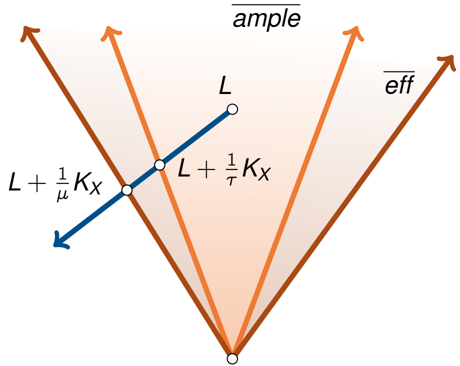
    $($ effective threshold $)^{-1}$
- $(\text{ample} \implies \text{big}) \implies \mu \leq \tau$

Algorithmic Discrete Mathematics

Polyhedral Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

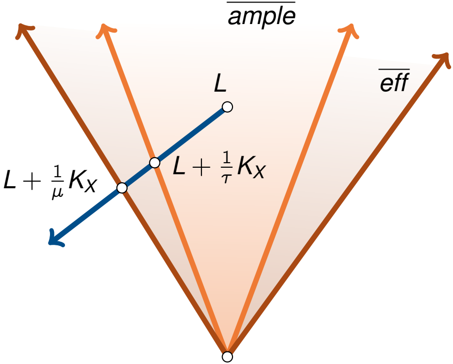
- nef-value: $\tau := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ ample }))^{-1}$
- unnormalized spectral value: $\mu := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ big }))^{-1}$



Polyhedral Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ ample}))^{-1}$
- unnormalized spectral value: $\mu := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ big}))^{-1}$



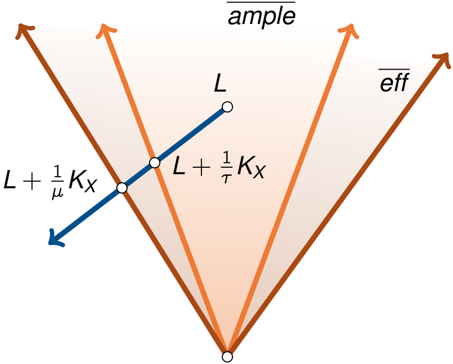
- ▷ $\tau \in \mathbb{Q}$
- ▷ $r \cdot K_X$ Cartier $\implies \tau \leq r(n+1)$.

[Kawamata]

Polyhedral Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \left(\sup\left(c \in \mathbb{R} \mid L + c \cdot K_X \text{ ample}\right)\right)^{-1}$
- unnormalized spectral value: $\mu := \left(\sup\left(c \in \mathbb{R} \mid L + c \cdot K_X \text{ big}\right)\right)^{-1}$

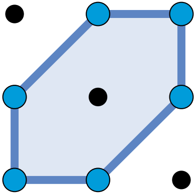


- $\tau \in \mathbb{Q}$
- $r \cdot K_X$ Cartier $\implies \tau \le r(n+1)$.

  [Kawamata]
- $\mu \le n+1$ [Beltrametti, Sommese '95]

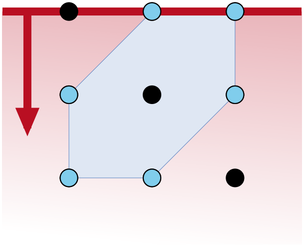
Lattice Polytopes

- lattice polytope: $P := \mathrm{conv}(v_1, \dots, v_k)$ for $v_j \in \mathbb{Z}^n$.



Lattice Polytopes

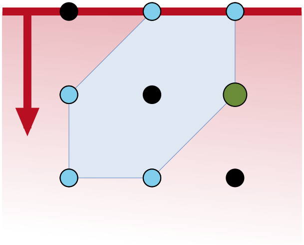
- lattice polytope : $P := \operatorname{conv}(v_1, \dots, v_k)$ for $v_j \in \mathbb{Z}^n$.
  - $P = \{x \mid Mx \leq b\}$ for M, b integral

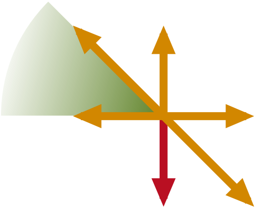


Lattice Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

- lattice polytope : $P := \text{conv}(v_1, \dots, v_k)$ for $v_j \in \mathbb{Z}^n$.
  - $P = \{x \mid Mx \leq b\}$ for M, b integral
  - normal fan $\text{NF}(P) \subseteq \mathbb{R}^d$

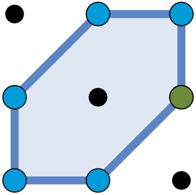


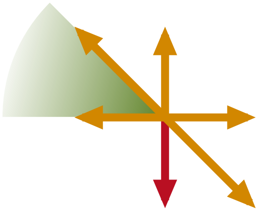


Lattice Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

- lattice polytope: $P := \mathrm{conv}(v_1, \dots, v_k)$ for $v_j \in \mathbb{Z}^n$.
  - $P = \{x \mid Mx \leq b\}$ for M, b integral
  - normal fan $\mathrm{NF}(P) \subseteq \mathbb{R}^d$
- complete polyhedral fan Σ $\longleftrightarrow$ projective toric variety X_Σ
  - for $\sigma \in \Sigma$ glue $\mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n])$ at common faces





Polyhedral Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- $P \subseteq \mathbb{R}^n$ lattice polytope, $\dim P = n$

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_i^t \mathbf{x} \leq b_i \,, \quad 1 \leq i \leq m\},$$

$\mathbf{a}_i \in \mathbb{Z}^n$ primitive, $b_i \in \mathbb{Z}$

Polyhedral Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

▷ $P \subseteq \mathbb{R}^n$ lattice polytope, $\dim P = n$

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_i^t \mathbf{x} \le b_i\,, \quad 1 \le i \le m\},$$

$\mathbf{a}_i \in \mathbb{Z}^n$ primitive, $b_i \in \mathbb{Z}$

▷ $c \geq 0$

▶ adjoint polytope $P^{(c)}$:

all points with lattice distance

$\geq c$ from each facet.

Polyhedral Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

▷ $P \subseteq \mathbb{R}^n$ lattice polytope, $\dim P = n$

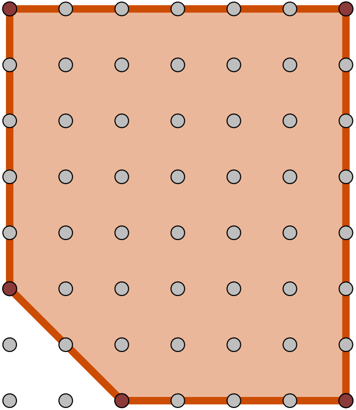
$$P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_i^t \mathbf{x} \leq b_i\,, \quad 1 \leq i \leq m\},$$

$\mathbf{a}_i \in \mathbb{Z}^n$ primitive, $b_i \in \mathbb{Z}$

▷ $c \geq 0$

▶ adjoint polytope $P^{(c)}$:

all points with lattice distance $\geq c$ from each facet.



▶ $c = 0$

Polyhedral Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

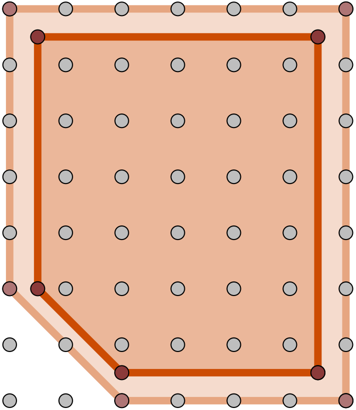
- $P \subseteq \mathbb{R}^n$ lattice polytope, $\dim P = n$

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_i^t \mathbf{x} \leq b_i\,, \quad 1 \leq i \leq m\},$$

$$\mathbf{a}_i \in \mathbb{Z}^n \text{ primitive}, \; b_i \in \mathbb{Z}$$

- $c \geq 0$
- adjoint polytope $P^{(c)}$:

  all points with lattice distance $\geq c$ from each facet.



- $c = \frac{1}{2}$

Polyhedral Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

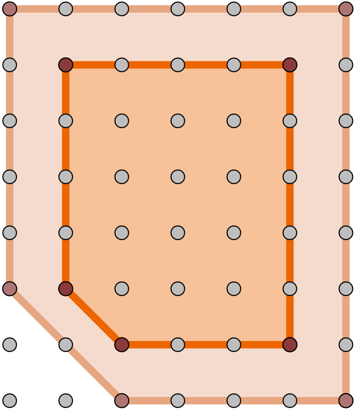
- ▷ $P \subseteq \mathbb{R}^n$ lattice polytope, $\dim P = n$

  $$P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_i^t \mathbf{x} \le b_i\,, \quad 1 \le i \le m\},$$

  $\mathbf{a}_i \in \mathbb{Z}^n$ primitive, $b_i \in \mathbb{Z}$

- ▷ $c \ge 0$
- ▶ adjoint polytope $P^{(c)}$:

  all points with lattice distance $\ge c$ from each facet.



▶ $c = 1$

Polyhedral Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

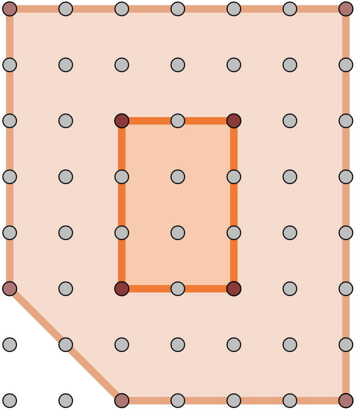
- $P \subseteq \mathbb{R}^n$ lattice polytope, $\dim P = n$

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_i^t \mathbf{x} \le b_i\,, \quad 1 \le i \le m\},$$

$\mathbf{a}_i \in \mathbb{Z}^n$ primitive, $b_i \in \mathbb{Z}$

- $c \ge 0$
- adjoint polytope $P^{(c)}$:

  all points with lattice distance $\ge c$ from each facet.



- $c = 2$

Polyhedral Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

▷ $P \subseteq \mathbb{R}^n$ lattice polytope, $\dim P = n$

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_i^t \mathbf{x} \le b_i\,, \quad 1 \le i \le m\},$$

$\mathbf{a}_i \in \mathbb{Z}^n$ primitive, $b_i \in \mathbb{Z}$

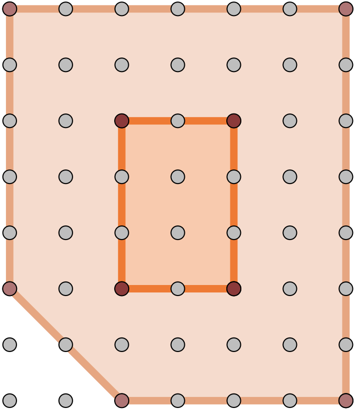
▷ $c \ge 0$

▶ adjoint polytope $P^{(c)}$:

all points with lattice distance $\ge c$ from each facet.

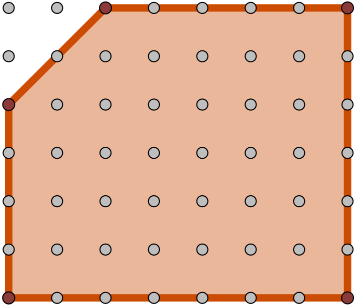
$$P^{(c)} = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{c} \mathbf{a}_i^t \mathbf{x} \le b_i - c\,, \\ 1 \le i \le m \end{array} \right\}$$

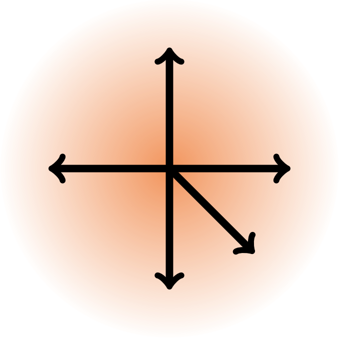
$$= \{\mathbf{x} \in \mathbb{R}^n \mid Ax \le b - c \cdot \mathbf{1}\}$$



▶ $c = 2$

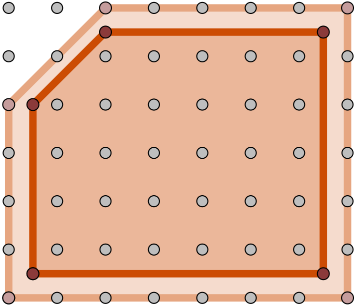
Invariants for Lattice Polytopes

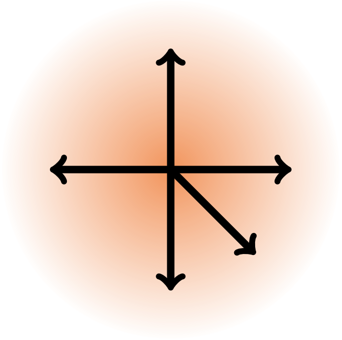




- $c = 0$
- $\tau =$
- $\mu =$

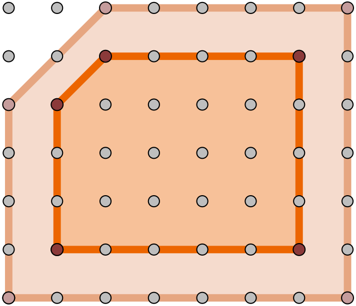
Invariants for Lattice Polytopes

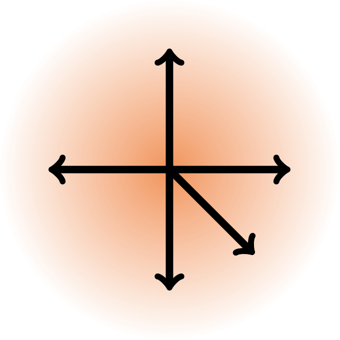




- $c = \frac{1}{2}$
- $\tau =$
- $\mu =$

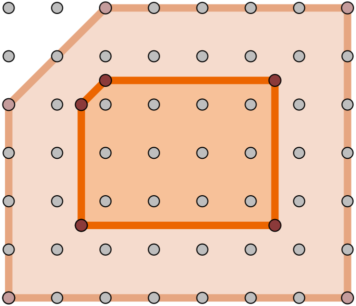
Invariants for Lattice Polytopes

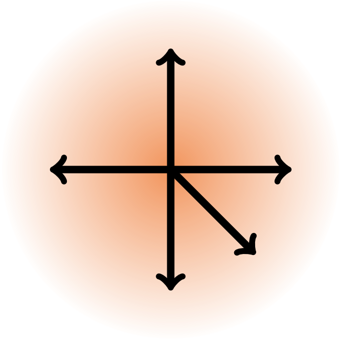




- $c = 1$
- $\tau =$
- $\mu =$

Invariants for Lattice Polytopes

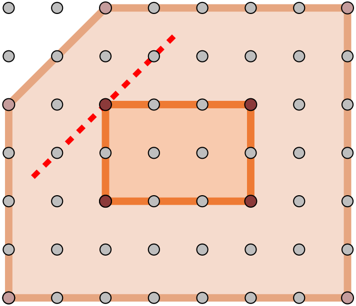


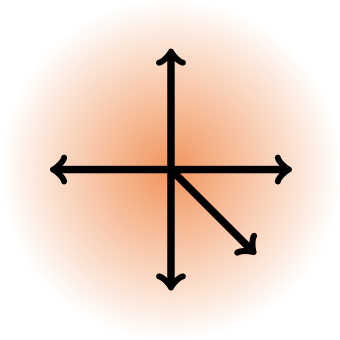


- $c = \frac{3}{2}$
- $\tau =$
- $\mu =$

Invariants for Lattice Polytopes

TECHNISCHE
UNIVERSITÄT
DARMSTADT



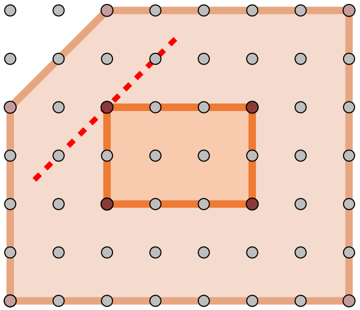


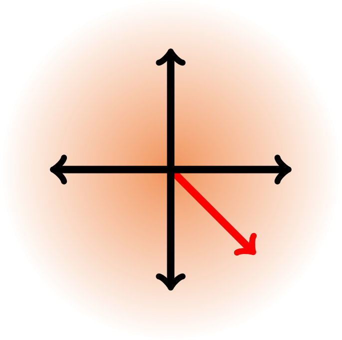
- $c = 2$
- $\tau =$
- $\mu =$

Invariants for Lattice Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

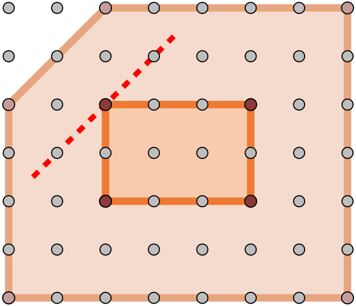
- nef-value: $\tau \; := \; \inf \left(c > 0 \;\middle|\; P \text{ and } P^{(c)} \text{ are combinatorially different } \right)^{-1}$

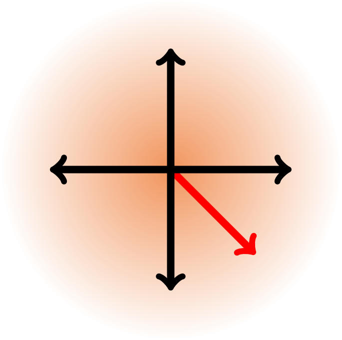




- $c = 2$
- $\tau = \frac{1}{2}$
- $\mu =$

- nef-value: $\tau \;:=\; \sup(\, c \;>0 \;\mid\; \mathrm{NF}(P) \;=\; \mathrm{NF}(P^{(c)})\;)^{-1}$



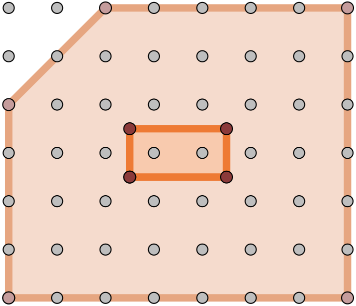


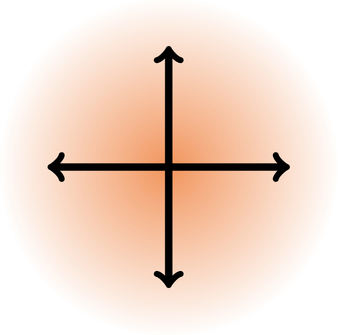
- $c = 2$
- $\tau = \frac{1}{2}$
- $\mu =$

Invariants for Lattice Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)}))^{-1}$



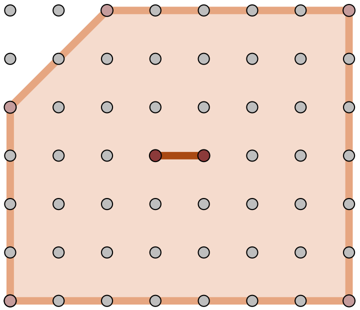


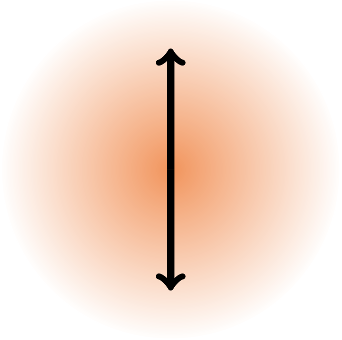
- $c = \frac{5}{2}$
- $\tau = \frac{1}{2}$
- $\mu =$

Invariants for Lattice Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)}))^{-1}$

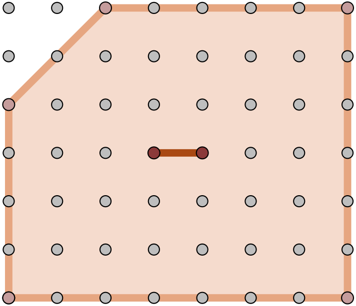


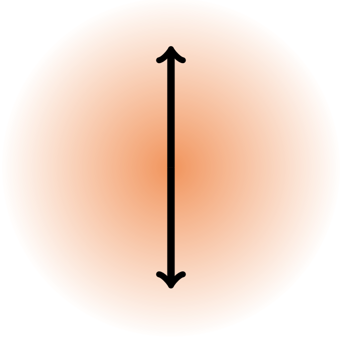


- $c = 3$
- $\tau = \frac{1}{2}$
- $\mu =$

Invariants for Lattice Polytopes

- nef-value: $\tau := \sup(c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)}))^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)} \neq \varnothing)^{-1}$

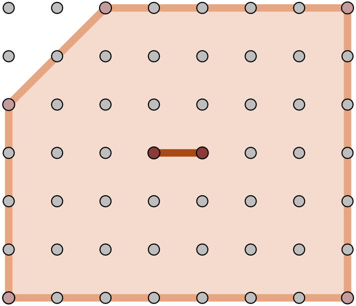


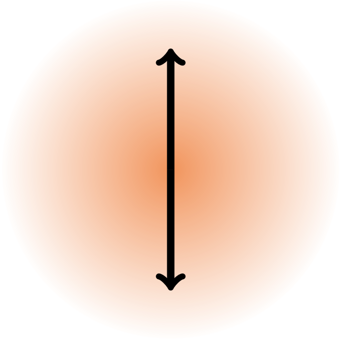


- $c = 3$
- $\tau = \frac{1}{2}$
- $\mu = \frac{1}{3}$

Invariants for Lattice Polytopes

- nef-value: $\tau := \sup(c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)}))^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)}$ is full-dimensional $)^{-1}$



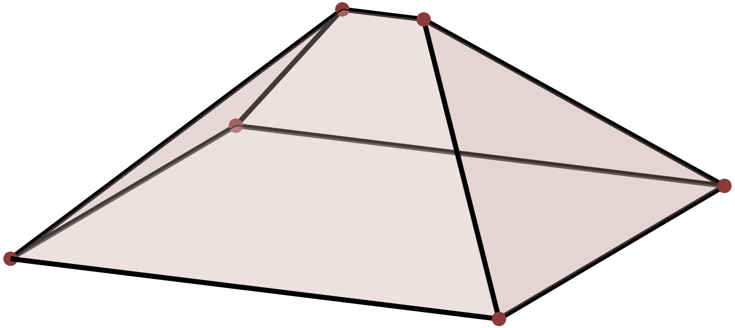


- $c = 3$
- $\tau = \frac{1}{2}$
- $\mu = \frac{1}{3}$

Invariants for Lattice Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)}))^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)}$ is full-dimensional $)^{-1}$

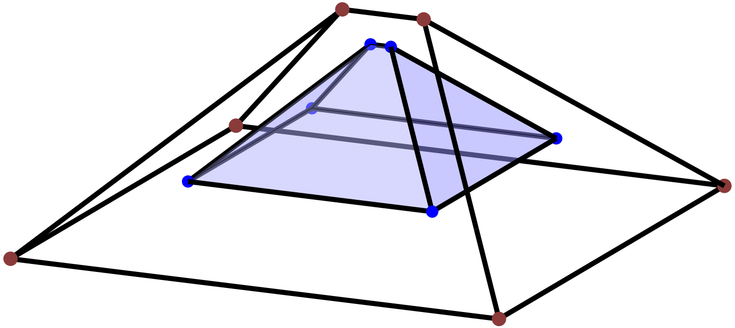


- $c = 0$
- $\tau =$
- $\mu =$

Invariants for Lattice Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

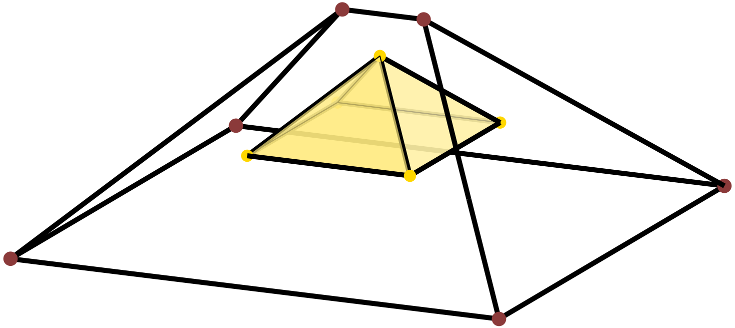
- nef-value: $\tau := \sup(c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)}))^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)}$ is full-dimensional $)^{-1}$



- $c = \frac{3}{2}$
- $\tau =$
- $\mu =$

Invariants for Lattice Polytopes

- nef-value: $\tau := \sup(c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)}))^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)} \text{ is full-dimensional})^{-1}$

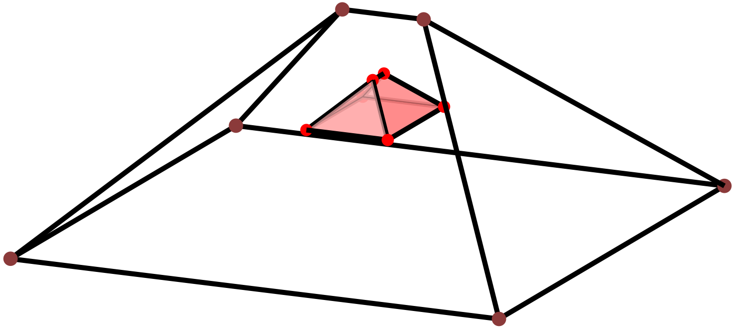


- $c = 2$
- $\tau = \frac{1}{2}$
- $\mu =$

Invariants for Lattice Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(\, c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)})\,)^{-1}$
- spectral value: $\mu := \sup(\, c \geq 0 \mid P^{(c)}$ is full-dimensional $)^{-1}$

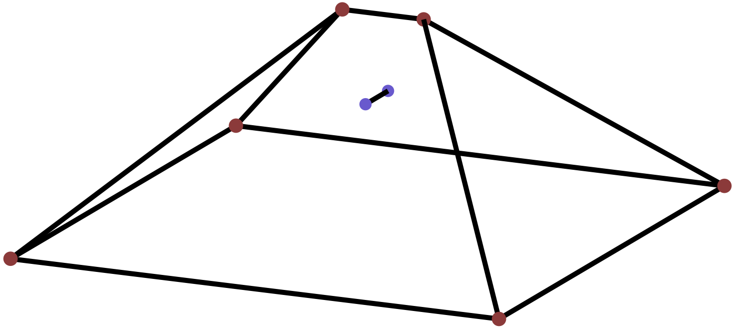


- $c = \frac{5}{2}$
- $\tau = \frac{1}{2}$
- $\mu =$

Invariants for Lattice Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)}))^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)}$ is full-dimensional $)^{-1}$



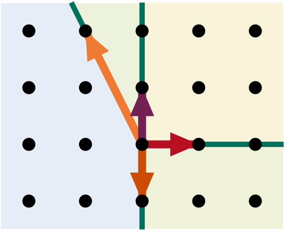
- $c = 3$
- $\tau = \frac{1}{2}$
- $\mu = \frac{1}{3}$

Lattice Polytopes and Toric Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

complete polyhedral fan $\Sigma_X \subseteq \mathbb{R}^n$ $\longleftrightarrow$ projective toric variety X_Σ

ray generators $u_i \in \mathbb{Z}^n$ $\longleftrightarrow$ torus invariant prime divisors D_i



Lattice Polytopes and Toric Varieties

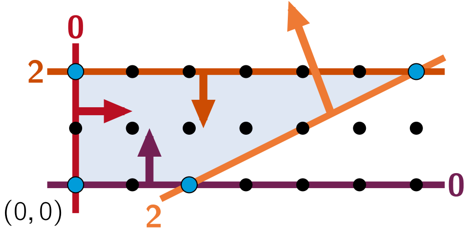
TECHNISCHE UNIVERSITÄT DARMSTADT

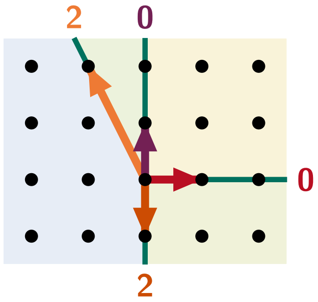
complete polyhedral fan $\Sigma_X \subseteq \mathbb{R}^n$ $\longleftrightarrow$ projective toric variety X_Σ

ray generators $u_i \in \mathbb{Z}^n$ $\longleftrightarrow$ torus invariant prime divisors D_i

lattice polytope $\longleftrightarrow$ polarized toric variety

$$P := \{x \in \mathbb{R}^d \mid \langle v, u_i\rangle \geq -a_i\} \qquad (X_P, \mathcal{O}_X(\textstyle\sum a_i D_i))$$





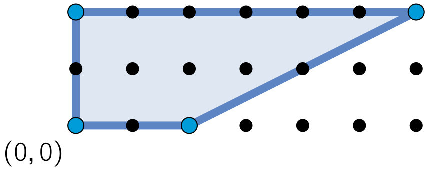
Lattice Polytopes and Toric Varieties

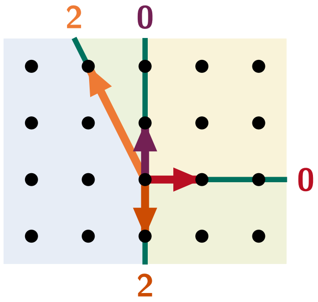
complete polyhedral fan $\Sigma_X \subseteq \mathbb{R}^n$ $\longleftrightarrow$ projective toric variety X_Σ

ray generators $u_i \in \mathbb{Z}^n$ $\longleftrightarrow$ torus invariant prime divisors D_i

lattice polytope $\longleftrightarrow$ polarized toric variety

$$P := \{x \in \mathbb{R}^d \mid \langle v, u_i \rangle \geq -a_i\} \qquad (X_P, \mathcal{O}_X(\sum a_i D_i))$$





Correspondence to Toric Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \text{sup}(c > 0 \mid \text{NF}(P) = \text{NF}(P^{(c)}))^{-1}$
- spectral value: $\mu := \text{sup}(c \geq 0 \mid P^{(c)} \text{ is full-dimensional})^{-1}$

Correspondence to Toric Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)}))^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)}$ is full-dimensional $)^{-1}$

▷ (X, L) polarized toric variety

▷ $\mathbb{Q}$-Cartier ▷ L ample ▷ Σ associated fan with rays

divisor D on X is ample $\longleftrightarrow$ $\mathrm{NF}(P_D) = \Sigma$

Correspondence to Toric Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(c > 0 \mid L + c \cdot K_X$ is ample $)^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)}$ is full-dimensional $)^{-1}$

- (X, L) polarized toric variety
  - $\mathbb{Q}$-Cartier
  - L ample
  - Σ associated fan with rays

divisor D on X is ample $\longleftrightarrow$ $\mathrm{NF}(P_D) = \Sigma$

Correspondence to Toric Varieties

- nef-value: $\tau := \sup(c > 0 \mid L + c \cdot K_X \text{ is ample })^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)} \text{ is full-dimensional})^{-1}$

▷ (X, L) polarized toric variety

▷ $\mathbb{Q}$-Cartier ▷ L ample ▷ Σ associated fan with rays

divisor D on X is ample $\longleftrightarrow$ $\mathrm{NF}(P_D) = \Sigma$

divisor D on X is big $\longleftrightarrow$ $\dim P_D = \dim X$

Correspondence to Toric Varieties

- nef-value: $\tau \;:=\; \sup(\, c \;>0 \;|\; L + c\cdot K_X \text{ is ample })^{-1}$
- spectral value: $\mu \;:=\; \sup(\, c \;\geq\; 0 \;|\; L + c\cdot K_X \text{ big })^{-1}$

- (X, L) polarized toric variety
  - $\mathbb{Q}$-Cartier
  - L ample
  - Σ associated fan with rays

$$\text{divisor } D \text{ on } X \text{ is ample} \quad\longleftrightarrow\quad \mathrm{NF}(P_D) = \Sigma$$

$$\text{divisor } D \text{ on } X \text{ is big} \quad\longleftrightarrow\quad \dim P_D = \dim X$$

Correspondence to Toric Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(c > 0 \mid L + c \cdot K_X \text{ is ample})^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid L + c \cdot K_X \text{ big})^{-1}$

▷ (X, L) polarized toric variety

▷ $\mathbb{Q}$-Cartier ▷ L ample ▷ Σ associated fan with rays

divisor D on X is ample $\longleftrightarrow$ $\mathrm{NF}(P_D) = \Sigma$

divisor D on X is big $\longleftrightarrow$ $\dim P_D = \dim X$

X $\mathbb{Q}$-Gorenstein $\longleftrightarrow$ K_X $\mathbb{Q}$-Cartier $\Longleftrightarrow$ $\tau > 0$

Correspondence to Toric Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(c > 0 \mid L + c \cdot K_X \text{ is ample })^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid L + c \cdot K_X \text{ big })^{-1}$

- (X, L) polarized toric variety
  - $\mathbb{Q}$-Cartier
  - L ample
  - Σ associated fan with rays

divisor D on X is ample $\longleftrightarrow$ $\mathrm{NF}(P_D) = \Sigma$

divisor D on X is big $\longleftrightarrow$ $\dim P_D = \dim X$

X $\mathbb{Q}$-Gorenstein $\longleftrightarrow$ K_X $\mathbb{Q}$-Cartier $\Longleftrightarrow$ $\tau > 0$

- let $0^{-1} = \infty$: $\Longrightarrow$ polyhedral definition does not need $\mathbb{Q}$-Gorenstein

Correspondence to Toric Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(c > 0 \mid L + c \cdot K_X \text{ is ample })^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid L + c \cdot K_X \text{ big })^{-1}$

- (X, L) polarized toric variety
  - $\mathbb{Q}$-Cartier
  - L ample
  - Σ associated fan with rays

divisor D on X is ample $\longleftrightarrow$ $\mathrm{NF}(P_D) = \Sigma$

divisor D on X is big $\longleftrightarrow$ $\dim P_D = \dim X$

X $\mathbb{Q}$-Gorenstein $\longleftrightarrow$ K_X $\mathbb{Q}$-Cartier $\iff$ $\tau > 0$

- let $0^{-1} = \infty$: $\implies$ polyhedral definition does not need $\mathbb{Q}$-Gorenstein

Polyhedral Adjunction Theory $\supsetneq$ **classical Adjunction Theory**

Computational Adjunction Theory

- combinatorial definition allows efficient computations

Computational Adjunction Theory

- combinatorial definition allows efficient computations
- `polymake`: computations in polyhedral geometry and related fields

Computational Adjunction Theory

- combinatorial definition allows efficient computations
- `polymake`: computations in polyhedral geometry and related fields
  - lattice polytopes, toric geometry, tropical geometry, ...
  - fully programmable interface: `perl`, C++
  - symmetric polyhedra, sage integration, GAP/Singular interface
- http://polymake.org

Computational Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- combinatorial definition allows efficient computations
- `polymake`: computations in polyhedral geometry and related fields
  - lattice polytopes, toric geometry, tropical geometry, ...
  - fully programmable interface: `perl`, C++
  - symmetric polyhedra, sage integration, GAP/Singular interface
- `http://polymake.org`
- `polymake` extension

  PolyhedralAdjunction `http://polymake.org/polytopes/paffenholz/data/polymake/extensions/PolyhedralAdjunction`

Computational Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- combinatorial definition allows efficient computations
- `polymake`: computations in polyhedral geometry and related fields
  - lattice polytopes, toric geometry, tropical geometry, ...
  - fully programmable interface: `perl`, `C++`
  - symmetric polyhedra, sage integration, GAP/Singular interface
- http://polymake.org
- `polymake` extension

  PolyhedralAdjunction http://polymake.org/polytopes/paffenholz/data/polymake/extensions/PolyhedralAdjunction
- `polymake` **workshop: March 22 and 23 in Darmstadt**

Cayley Sums

TECHNISCHE UNIVERSITÄT DARMSTADT

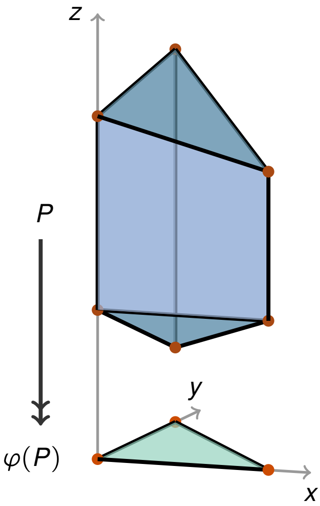
- P n-dimensional lattice polytope
- P is a Cayley sum

$$:\Longleftrightarrow$$

P has a projection onto standard simplex

$$\varphi : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^m \quad \text{lattice projection}$$

such that $\quad \varphi(P) = \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_m).$



Cayley Sums

TECHNISCHE UNIVERSITÄT DARMSTADT

- ▷ P n-dimensional lattice polytope
- ▶ P is a Cayley sum

$$:\Longleftrightarrow$$

P has a projection onto standard simplex

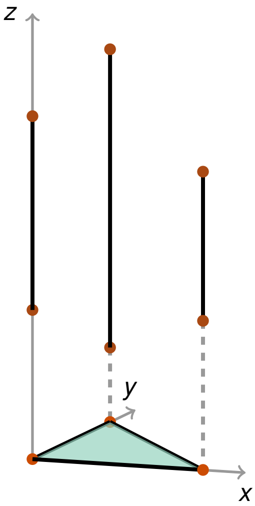
$$\varphi : \mathbb{R}^n \longrightarrow\!\!\!\!\rightarrow \mathbb{R}^m \quad \text{lattice projection}$$

such that $\quad \varphi(P) = \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_m).$

- ▷ equivalently

$$P = \operatorname{conv}(P_0 \times \{\mathbf{0}\}, P_1 \times \{\mathbf{e}_1\}, \dots, P_m \times \{\mathbf{e}_m\})$$

for lattice polytopes $P_0, \dots, P_m \in \mathbb{R}^{n-m}$.



Cayley Sums

TECHNISCHE UNIVERSITÄT DARMSTADT

- ▷ P n-dimensional lattice polytope
- ▶ P is a Cayley sum

$$:\Longleftrightarrow$$

P has a projection onto standard simplex

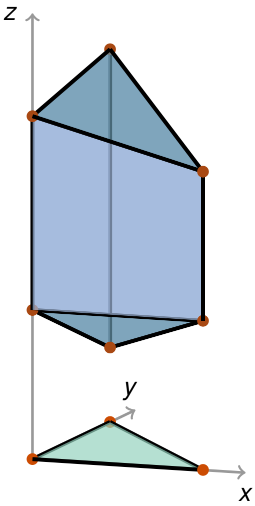
$$\varphi : \mathbb{R}^n \longrightarrow\!\!\!\!\!\rightarrow \mathbb{R}^m \quad \text{lattice projection}$$

$$\text{such that} \quad \varphi(P) = \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_m).$$

- ▷ equivalently

$$P = \operatorname{conv}(P_0 \times \{\mathbf{0}\}, P_1 \times \{\mathbf{e}_1\}, \dots, P_m \times \{\mathbf{e}_m\})$$

for lattice polytopes $P_0, \dots, P_m \in \mathbb{R}^{n-m}$.



Large spectral value implies Cayley

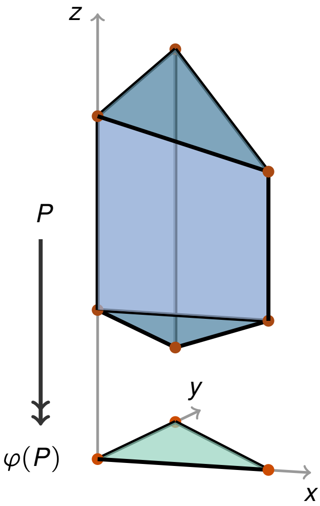
TECHNISCHE UNIVERSITÄT DARMSTADT

- P n-dimensional lattice polytope
- Theorem

  $\mu \geq \frac{n+2}{2} \implies P$ is a Cayley sum

  of lattice polytopes in R^m

  $$m \leq \lfloor 2(n+1-\mu) \rfloor$$



Large spectral value implies Cayley

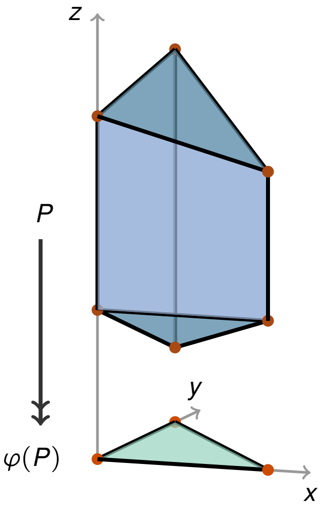
- P n-dimensional lattice polytope

- **Theorem**

  $\mu \geq \frac{n+2}{2} \implies P$ is a Cayley sum of lattice polytopes in R^m

  $$m \leq \lfloor 2(n+1-\mu) \rfloor$$

  - $\mu \geq \frac{n+2}{2} \iff n > m$



Large spectral value implies Cayley

TECHNISCHE UNIVERSITÄT DARMSTADT

- ▷ P n-dimensional lattice polytope
- ▶ Theorem

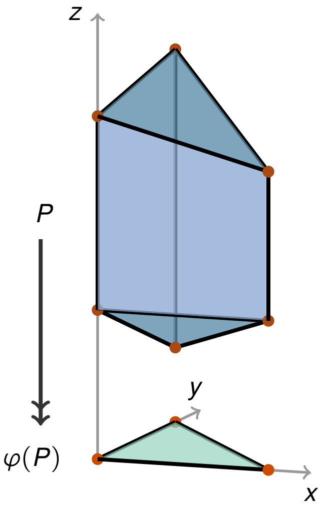
  $\mu \geq \frac{n+2}{2} \implies P$ is a Cayley sum

  of lattice polytopes in R^m

  $m \leq \lfloor 2(n+1-\mu) \rfloor$

  - ▷ $\mu \geq \frac{n+2}{2} \iff n > m$
- ▶ Corollary

  $\mu \geq \frac{n+2}{2} \implies P$ has lattice width one



Large spectral value implies Cayley

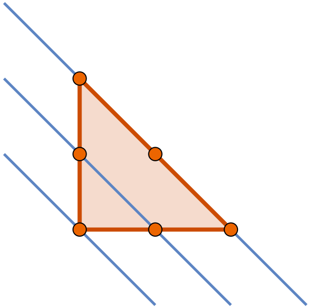
TECHNISCHE UNIVERSITÄT DARMSTADT

- P n-dimensional lattice polytope

- **Theorem**

  $$\mu \geq \frac{n+2}{2} \implies P \text{ is a Cayley sum of lattice polytopes in } R^m$$
  $$m \leq \lfloor 2(n+1-\mu) \rfloor$$

  - $\mu \geq \frac{n+2}{2} \iff n > m$



- **Corollary**

  $$\mu \geq \frac{n+2}{2} \implies P \text{ has lattice width one}$$

- best possible bound:

  $$X = (\mathbb{P}^2, O(2)) \rightsquigarrow P = 2\Delta_2, \ \mu = \frac{n+1}{2}$$

Large spectral value implies Cayley

TECHNISCHE UNIVERSITÄT DARMSTADT

- P n-dimensional lattice polytope
- Theorem

  $\mu \geq \frac{n+2}{2} \implies$ morphism $\pi : \mathbb{P}(H_0 \oplus H_1 \oplus \cdots \oplus H_m) \longrightarrow X$

  H_i: line bundles

  on toric variety in dimension $\leq 2(n+1-\mu)$

  - $\mu \geq \frac{n+2}{2} \iff n > m$
- Corollary

  $\mu \geq \frac{n+2}{2} \implies P$ has lattice width one
- best possible bound:

  $X = (\mathbb{P}^2, O(2)) \rightsquigarrow P = 2\Delta_2\,,\ \mu = \frac{n+1}{2}$

Relation to Ehrhart Theory

- Ehrhart Theory:
  - Count lattice points in a polytope P: $|P \cap \mathbb{Z}^n|$.

Relation to Ehrhart Theory

- Ehrhart Theory:
  - Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.

Relation to Ehrhart Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Ehrhart Theory:
  - Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\operatorname{codeg} P := \min\left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing\right)$

Algorithmic Discrete Mathematics

Relation to Ehrhart Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Ehrhart Theory:
  - ▹ Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\operatorname{codeg} P := \min\left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing\right)$
    - ▹ $\operatorname{codeg} P \leq n + 1$ with equality only for a unimodular simplex

Algorithmic Discrete Mathematics

Relation to Ehrhart Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Ehrhart Theory:
  - Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\operatorname{codeg} P := \min\left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing\right)$
    - $\operatorname{codeg} P \leq n+1$ with equality only for a unimodular simplex
- $\mu \leq \operatorname{codeg} P$:

Relation to Ehrhart Theory

- Ehrhart Theory:
  - Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\operatorname{codeg} P := \min\left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing \right)$
    - $\operatorname{codeg} P \leq n+1$ with equality only for a unimodular simplex
- $\mu \leq \operatorname{codeg} P$: $\operatorname{int}(kP) \cap \mathbb{Z}^n \subseteq (kP)^{(1)}$

Relation to Ehrhart Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Ehrhart Theory:
  - ▷ Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\operatorname{codeg} P := \min\left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing\right)$
    - ▷ $\operatorname{codeg} P \leq n+1$ with equality only for a unimodular simplex
- $\mu \leq \operatorname{codeg} P$: $\operatorname{int}(kP) \cap \mathbb{Z}^n \subseteq (kP)^{(1)} = k\left(P^{(\frac{1}{k})}\right)$

Relation to Ehrhart Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Ehrhart Theory:
  - Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\operatorname{codeg} P := \min\left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing \right)$
    - $\operatorname{codeg} P \leq n+1$ with equality only for a unimodular simplex
- $\mu \leq \operatorname{codeg} P$: $\operatorname{int}(kP) \cap \mathbb{Z}^n \subseteq (kP)^{(1)} = k\left(P^{(\frac{1}{k})}\right)$
- Cayley conjecture

  $\operatorname{codeg} P > \frac{n+2}{2} \Rightarrow P$ has lattice width one [Batyrev, Nill], [. . .]

Relation to Ehrhart Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Ehrhart Theory:
  - ▷ Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\operatorname{codeg} P := \min\left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing\right)$
    - ▷ $\operatorname{codeg} P \leq n+1$ with equality only for a unimodular simplex
- $\mu \leq \operatorname{codeg} P$: $\operatorname{int}(kP) \cap \mathbb{Z}^n \subseteq (kP)^{(1)} = k\left(P^{(\frac{1}{k})}\right)$
- Cayley conjecture

  $\operatorname{codeg} P > \frac{n+2}{2} \Rightarrow P$ has lattice width one [Batyrev, Nill], [. . .]

  - ▷ X_P smooth and $\mu = \tau$ [Dickenstein, DiRocco, Piene]

Relation to Ehrhart Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Ehrhart Theory:
  - Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\operatorname{codeg} P := \min\left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing\right)$
    - $\operatorname{codeg} P \leq n + 1$ with equality only for a unimodular simplex
- $\mu \leq \operatorname{codeg} P$: $\operatorname{int}(kP) \cap \mathbb{Z}^n \subseteq (kP)^{(1)} = k\left(P^{(\frac{1}{k})}\right)$
- Cayley conjecture

  $\operatorname{codeg} P > \frac{n+2}{2} \Rightarrow P$ has lattice width one [Batyrev, Nill], [. . .]

  - X_P smooth and $\mu = \tau$ [Dickenstein, DiRocco, Piene]
  - X_P smooth [Dickenstein, Nill]

Relation to Ehrhart Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Ehrhart Theory:
  - Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\operatorname{codeg} P := \min\left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing\right)$
    - $\operatorname{codeg} P \leq n+1$ with equality only for a unimodular simplex
- $\mu \leq \operatorname{codeg} P$: $\operatorname{int}(kP) \cap \mathbb{Z}^n \subseteq (kP)^{(1)} = k\left(P^{(\frac{1}{k})}\right)$
- Cayley conjecture

$$\operatorname{codeg} P > \frac{n+2}{2} \Rightarrow P \text{ has lattice width one}$$ [Batyrev, Nill], [. . .]

  - X_P smooth and $\mu = \tau$ [Dickenstein, DiRocco, Piene]
  - X_P smooth [Dickenstein, Nill]
  - here: X_P Gorenstein and $\mu = \tau$

Dual Defective Varieties

- (X_P, L_P) polarized toric variety with polytope P
- $X_P \hookrightarrow \mathbb{P}^{|A|-1}$ for $A := P \cap \mathbb{Z}^n$ projective embedding

Dual Defective Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- (X_P, L_P) polarized toric variety with polytope P
- $X_P \hookrightarrow \mathbb{P}^{|A|-1}$ for $A := P \cap \mathbb{Z}^n$ projective embedding

- dual variety $X_P^\vee := \overline{\left\{ H \in (\mathbb{P}^{m-1})^* \;\middle|\; H \text{ tangent to } X_P \text{ at some } x \in X_P \right\}}$

Dual Defective Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- (X_P, L_P) polarized toric variety with polytope P
- $X_P \hookrightarrow \mathbb{P}^{|A|-1}$ for $A := P \cap \mathbb{Z}^n$ projective embedding

- dual variety $X_P^\vee := \overline{\left\{ H \in (\mathbb{P}^{m-1})^* \;\middle|\; H \text{ tangent to } X_P \text{ at some } x \in X_P \right\}}$

- generically $X_P^\vee$ is a hypersurface
- X_P dual defective if $\dim X_P^\vee$ if $\dim X_P^\vee < |A|$

Dual Defective Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- (X_P, L_P) polarized toric variety with polytope P
- $X_P \hookrightarrow \mathbb{P}^{|A|-1}$ for $A := P \cap \mathbb{Z}^n$ projective embedding

- dual variety $X_P^\vee := \overline{\left\{H \in (\mathbb{P}^{m-1})^* \mid H \text{ tangent to } X_P \text{ at some } x \in X_P\right\}}$

- generically $X_P^\vee$ is a hypersurface
- X_P dual defective if $\dim X_P^\vee$ if $\dim X_P^\vee < |A|$

- Theorem: If $\mu \geq \frac{3n+4}{4}$ then X_A is dual defective

Dual Defective Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- ▷ (X_P, L_P) polarized toric variety with polytope P
- ▷ $X_P \hookrightarrow \mathbb{P}^{|A|-1}$ for $A := P \cap \mathbb{Z}^n$ projective embedding
- ▶ dual variety $X_P^\vee := \overline{\{H \in (\mathbb{P}^{m-1})^* \mid H \text{ tangent to } X_P \text{ at some } x \in X_P\}}$
- ▷ generically $X_P^\vee$ is a hypersurface
- ▶ X_P dual defective if $\dim X_P^\vee$ if $\dim X_P^\vee < |A|$
- ▶ Theorem: If $\mu \geq \frac{3n+4}{4}$ then X_A is dual defective
  - ▷ P Cayley polytope of order m in dimension $r > m$

    $\Longrightarrow X_P$ has dual defect $r - m$ [Dickenstein, Feichtner, Sturmfels]

Dual Defective Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

▷ (X_P, L_P) polarized toric variety with polytope P

▷ $X_P \longrightarrow \mathbb{P}^{|A|-1}$ for $A := P \cap \mathbb{Z}^n$ projective embedding

▶ dual variety $X_P^\vee := \overline{\left\{H \in (\mathbb{P}^{m-1})^* \mid H \text{ tangent to } X_P \text{ at some } x \in X_P\right\}}$

▷ generically $X_P^\vee$ is a hypersurface

▶ X_P dual defective if $\dim X_P^\vee$ if $\dim X_P^\vee < |A|$

▶ Theorem: If $\mu \geq \frac{3n+4}{4}$ then X_A is dual defective

▷ P Cayley polytope of order m in dimension $r > m$

$\Longrightarrow X_P$ has dual defect $r - m$ [Dickenstein, Feichtner, Sturmfels]

▶ Question Does $\mu > \frac{n+2}{2}$ suffice?

Proof Sketch

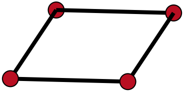
TECHNISCHE UNIVERSITÄT DARMSTADT

- Theorem $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m

Proof Sketch

TECHNISCHE UNIVERSITÄT DARMSTADT

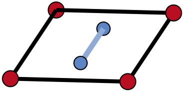
- Theorem $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m
  - let $\mu \geq \frac{n+2}{2}$, $\quad P = \{\mathbf{x} \mid \mathbf{a}_i^t\mathbf{x} \leq b_i\}$



Proof Sketch

TECHNISCHE UNIVERSITÄT DARMSTADT

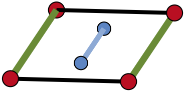
- Theorem $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m
  - let $\mu \geq \frac{n+2}{2}$, $\quad P = \{\mathbf{x} \mid \mathbf{a}_i^t \mathbf{x} \leq b_i\}$
  - core of P: $\text{core}(P) := P^{(\frac{1}{\mu})}$



Proof Sketch

TECHNISCHE UNIVERSITÄT DARMSTADT

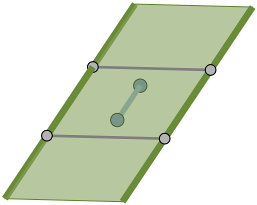
- Theorem $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m
  - let $\mu \geq \frac{n+2}{2}, \quad P = \{\mathbf{x} \mid \mathbf{a}_i^t \mathbf{x} \leq b_i\}$
  - core of P: $\operatorname{core}(P) := P^{(\frac{1}{\mu})}$
  - core facet: $\mathbf{a}_i^t \mathbf{x} + \mu \leq b_i$ supporting $\operatorname{aff} \operatorname{core}(P)$



Proof Sketch

TECHNISCHE UNIVERSITÄT DARMSTADT

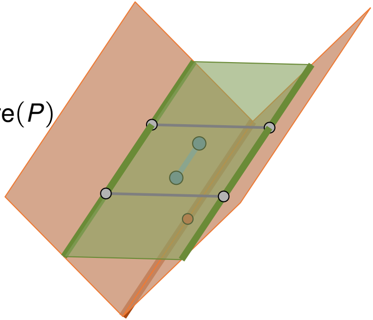
- Theorem $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m
  - let $\mu \geq \frac{n+2}{2}, \quad P = \{\mathbf{x} \mid \mathbf{a}_i^t\mathbf{x} \leq b_i\}$
  - core of P: $\operatorname{core}(P) := P^{(\frac{1}{\mu})}$
  - core facet: $\mathbf{a}_i^t\mathbf{x} + \mu \leq b_i$ supporting $\operatorname{aff}\operatorname{core}(P)$
  - Q: polyhedron spanned by core facets



Proof Sketch

TECHNISCHE UNIVERSITÄT DARMSTADT

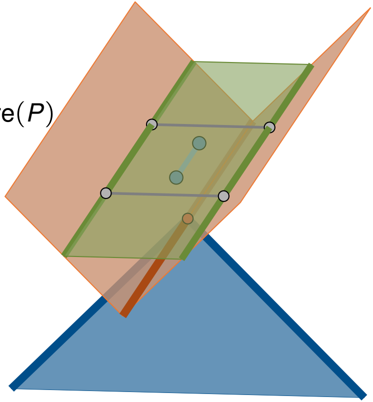
- **Theorem** $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m
  - let $\mu \geq \frac{n+2}{2}, \quad P = \{\mathbf{x} \mid \mathbf{a}_i^t\mathbf{x} \leq b_i\}$
  - core of P: $\operatorname{core}(P) := P^{(\frac{1}{\mu})}$
  - core facet: $\mathbf{a}_i^t\mathbf{x} + \mu \leq b_i$ supporting $\operatorname{aff}\operatorname{core}(P)$
  - Q: polyhedron spanned by core facets
  - $\sigma := Q \times \{1\}$



Proof Sketch

TECHNISCHE UNIVERSITÄT DARMSTADT

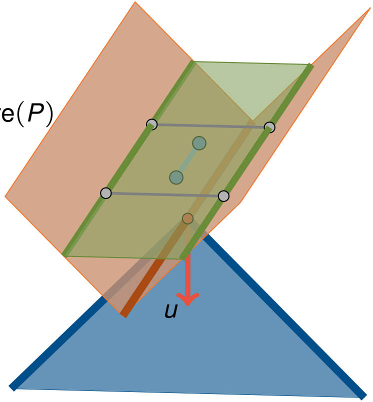
- **Theorem** $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m
  - let $\mu \geq \frac{n+2}{2}, \quad P = \{\mathbf{x} \mid \mathbf{a}_i^t\mathbf{x} \leq b_i\}$
  - core of P: $\mathrm{core}(P) := P^{(\frac{1}{\mu})}$
  - core facet: $\mathbf{a}_i^t\mathbf{x} + \mu \leq b_i$ supporting $\mathrm{aff\,core}(P)$
  - Q: polyhedron spanned by core facets
  - $\sigma := Q \times \{1\}$
  - $\sigma^\vee$: cone dual to $Q \times \{1\}$



Proof Sketch

TECHNISCHE UNIVERSITÄT DARMSTADT

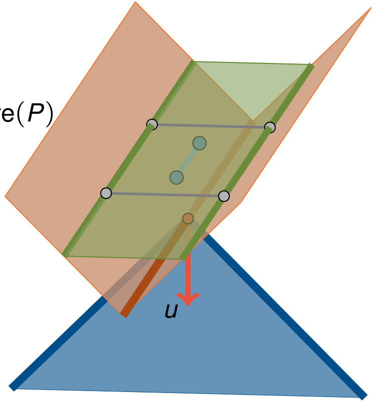
- Theorem $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m
  - let $\mu \geq \frac{n+2}{2}$, $\quad P = \{\mathbf{x} \mid \mathbf{a}_i^t\mathbf{x} \leq b_i\}$
  - core of P: $\text{core}(P) := P^{(\frac{1}{\mu})}$
  - core facet: $\mathbf{a}_i^t\mathbf{x} + \mu \leq b_i$ supporting $\operatorname{aff}\text{core}(P)$
  - Q: polyhedron spanned by core facets
  - $\sigma := Q \times \{1\}$
  - $\sigma^\vee$: cone dual to $Q \times \{1\}$
  - contains the functional $\mathbf{u} = (\mathbf{0}, 1)$.



Proof Sketch

TECHNISCHE
UNIVERSITÄT
DARMSTADT

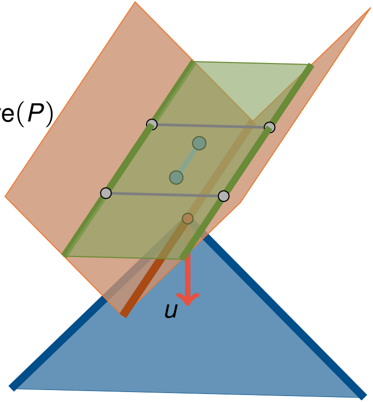
- ▶ Theorem $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m
  - ▷ let $\mu \geq \frac{n+2}{2}, \quad P = \{\mathbf{x} \mid \mathbf{a}_i^t\mathbf{x} \leq b_i\}$
  - ▷ core of P: $\operatorname{core}(P) := P^{(\frac{1}{\mu})}$
  - ▷ core facet: $\mathbf{a}_i^t\mathbf{x} + \mu \leq b_i$ supporting $\operatorname{aff}\operatorname{core}(P)$
  - ▷ Q: polyhedron spanned by core facets
  - ▷ $\sigma := Q \times \{1\}$
  - ▷ $\sigma^\vee$: cone dual to $Q \times \{1\}$
  - ▷ contains the functional $\mathbf{u} = (\mathbf{0}, 1)$.
  - ▷ show that $\mathbf{u}$ is sum of lattice points in σ



Proof Sketch

TECHNISCHE UNIVERSITÄT DARMSTADT

- Theorem $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m
  - let $\mu \geq \frac{n+2}{2}, \quad P = \{\mathbf{x} \mid \mathbf{a}_i^t\mathbf{x} \leq b_i\}$
  - core of P: $\mathrm{core}(P) := P^{(\frac{1}{\mu})}$
  - core facet: $\mathbf{a}_i^t\mathbf{x} + \mu \leq b_i$ supporting $\mathrm{aff}\,\mathrm{core}(P)$
  - Q: polyhedron spanned by core facets
  - $\sigma := Q \times \{1\}$
  - $\sigma^\vee$: cone dual to $Q \times \{1\}$
  - contains the functional $\mathbf{u} = (\mathbf{0}, 1)$.
  - show that $\mathbf{u}$ is sum of lattice points in σ
  - this is equivalent to P being Cayley



[Batyrev, Nill]

Computational Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- combinatorial definition allows efficient computations
- `polymake`: computations in polyhedral geometry and related fields
  - lattice polytopes, toric geometry, tropical geometry, ...
  - fully programmable interface: `perl`, C++
  - symmetric polyhedra, sage integration, GAP/Singular interface
- `http://polymake.org`
- `polymake` extension

  PolyhedralAdjunction `http://polymake.org/polytopes/paffenholz/data/polymake/extensions/PolyhedralAdjunction`
- `polymake` **workshop: March 22 and 23 in Darmstadt**