Permutation, Marginal, and Cut Polytopes

ISMP 2012, Berlin, Friday, August 24

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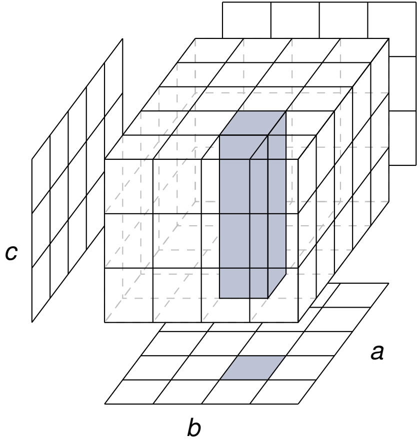

Andreas Paffenholz

joint with

Barbara Baumeister (Bielefeld)

Christian Haase (Frankfurt)

Benjamin Nill (Case Western)

arXiv:0709.1615, arXiv:1109.0191, arXiv:13??:????

Permutation Polytopes

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- S_n permutation group on $[n] := \{0, \ldots, n-1\}$
- $G \leqslant S_n$ sub-group

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- standard representation

$$g \in S_n \quad \longleftrightarrow \quad \text{permutation matrix} \quad M(g) = (m_{ij}) \in \mathbb{R}^{n \times n}$$

$$m_{ij} = \begin{cases} 1 & g(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

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- **Permutation polytope** $P(G) := \operatorname{conv}(M(g) \in \mathbb{R}^{n \times n} \mid g \in G)$
  - all $M(g)$ are vertices.
  - $P(G)$ is a $0/1$-polytope

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  - $P(G)$ is a $0/1$-polytope
- $B_n := P(S_n)$ Birkhoff polytope, doubly stochastic matrices [Billera&Sarangarajan 96], [Brualdi&Gibson 77]
  - $n!$ vertices and n^2 facets in dimension $(n-1)^2$

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$$P(G) = \operatorname{conv}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right) = \Delta_2$$

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Marginal Polytopes

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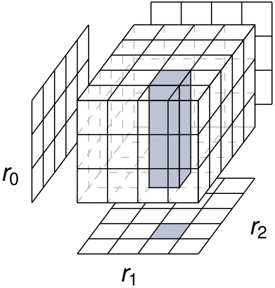
- k-way tables with side lengths $r_0, \dots, r_{k-1}$



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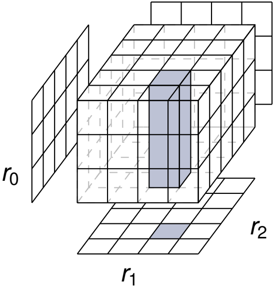

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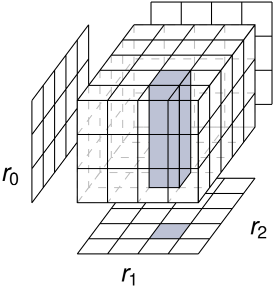

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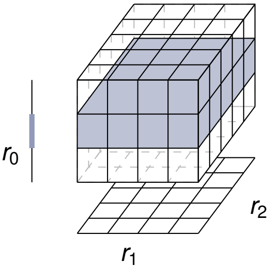

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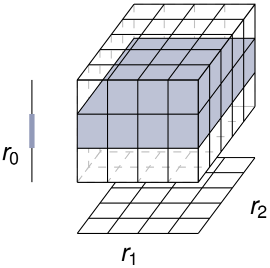
  maps std basis of table to margins



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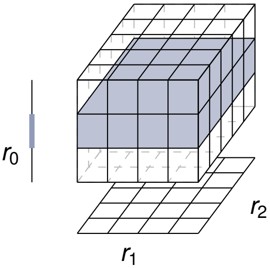
- marginal polytope

$$\operatorname{marg}_\Delta(r_0, \dots, r_{k-1}) := \operatorname{conv}(\operatorname{columns}(M))$$

$$:= \operatorname{conv}(Me_i \mid e_i \text{ std unit vector})$$



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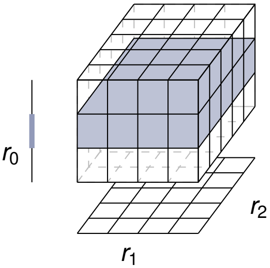

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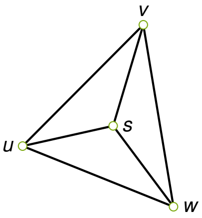
- applications in statistics
  - $\longrightarrow$ statistical disclosure limitation
  - $\longrightarrow$ linear/integer optimization over $\operatorname{marg}_\Delta(r_1, \ldots, r_k)$.

Cut Polytopes

- $\Gamma := (V, E)$ graph
- $S \subseteq V$ cut: $\delta(S) := \{uv \in E \mid u \in S,\ v \notin S\}$.



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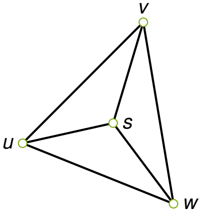

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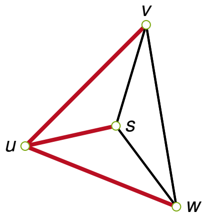
- example: $\mathrm{cut}(K_4) \ \cong \ \Delta_3 \oplus \Delta_3$
  - 6-dimensional polytope
  - 8 vertices and 16 facets.



uv	uw	vw	us	vs	ws
0	0	0	0	0	0
1	1	0	1	0	0
1	0	1	0	1	0
0	1	1	0	0	1
0	1	1	1	1	0
1	0	1	1	0	1
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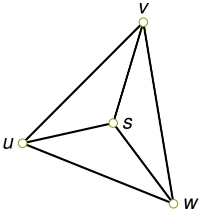
- (some) faces/facets
  - $0 \leqslant x_e \leqslant 1$
  - C cycle, $F \subset C$ with $|F|$ odd, then

    $$x(F) - x(C \setminus F) \leqslant |F| - 1$$

    cycle inequalities



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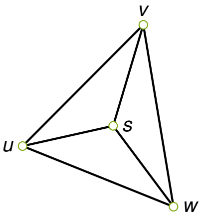

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  - Γ has no K_5-minor if and only if these suffice

[Barahona, Mahjoub]

Cut and Marginal Polytopes

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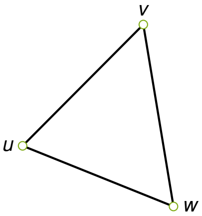
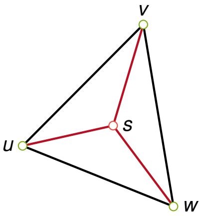


Cut and Marginal Polytopes

- $\Gamma = (V, E)$ graph
- $\Gamma \star s$: Add node s, connect to all other nodes.



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- ▷ $\Gamma = (V, E)$ graph
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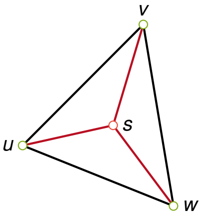
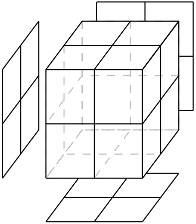
- ▶ $\text{marg}_\Gamma(2, \ldots, 2) \;\cong\; \text{cut}(\Gamma \star v)$





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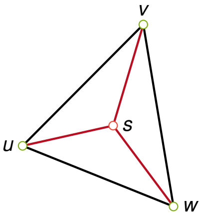

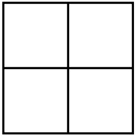

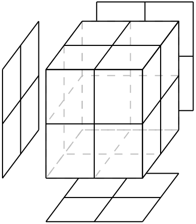

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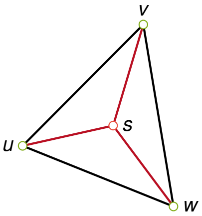

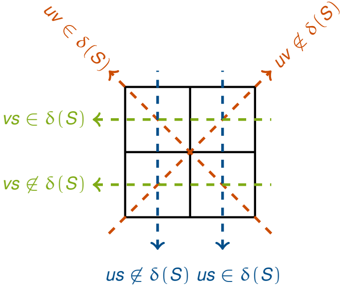

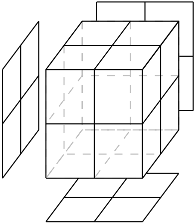

Permutation Polytopes: Example I

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- Example: $G = \langle\, (1\ 2)\,(3\ 4\ 5)\,\rangle \;\cong\; C_2 \times C_3$: $\quad P(G) = \Delta_1 \times \Delta_2$.

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix}\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\end{pmatrix}\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix}$$

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- **Theorem** [BHNP]

  $g = z_1 \cdot z_2$ disjoint cycles,

  - length t_1, t_2,
  - $\gcd(t_1, t_2) = 1$

  $\Longrightarrow P(G) = \Delta_{t_1-1} \times \Delta_{t_2-1}$

example

$G = \langle\, (0\ 1\ 2)(3\ 4)\,\rangle$

$P(G) = \Delta_2 \times \Delta_1$

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example

$G = \langle\, (0\ 1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9)\,\rangle$

$P(G) = (\Delta_2 \times \Delta_1) \star (\Delta_2 \times \Delta_1)$

Permutation Polytopes: Example II

- a, b, c pairwise relative prime
- $g = z_1 \cdot z_2 \cdot z_3$ disjoint cycles of lengths ab, ac, and bc
- $G := \langle g \rangle$

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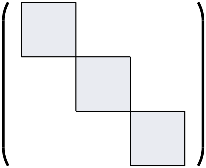
$$\begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}$$



Permutation Polytopes: Example II

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- a, b, c pairwise relative prime
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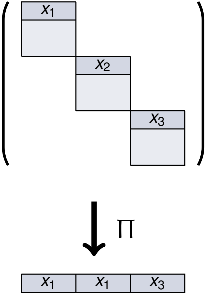

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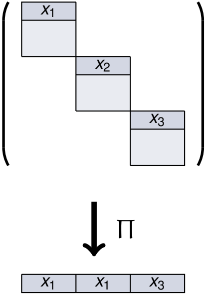

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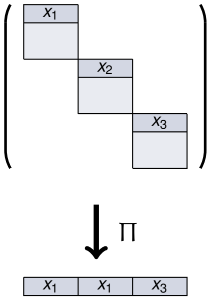
a	b	c	#dimension	#vertices	#facets
2	3	5	21	30	211
2	3	7	29	42	797
3	4	5	35	60	29387



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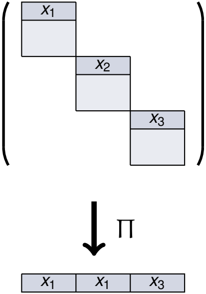

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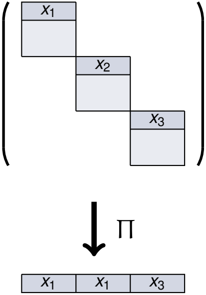

- for $P(a, b, c)$ in alg. statistics see e.g.

  http://markov-bases.de/ [Kahle&Rauh]

Marginal and Permutation Polytopes

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Theorem Marginal polytopes are permutation polytopes of abelian groups

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Theorem Marginal polytopes are permutation polytopes of abelian groups

- 2-way tables, $\Delta := \Delta(\{1\},\{2\})$

$$\rho : C_p \times C_q \curvearrowright \mathbb{R}^p \oplus \mathbb{R}^q,$$

$$(i,j) \longmapsto (e_i, e_j)$$

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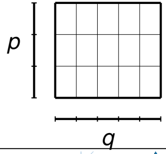
- restriction to first row in each block suffices

$\Longrightarrow$ corresponds to $(p \times q)$-tables

$\longrightarrow \quad \mathrm{marg}_\Delta(p,q) = P(G)$



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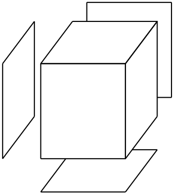

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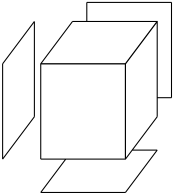

$$\rho : C_p \times C_q \times C_r \curvearrowright \mathbb{R}^{pq} \oplus \mathbb{R}^{pr} \oplus \mathbb{R}^{qr}$$

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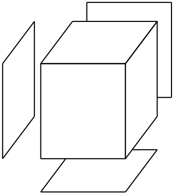

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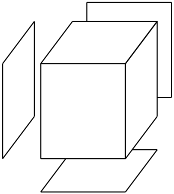

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- again first row suffices: $e_{0,0} \mapsto e_{i,j}$ determines map

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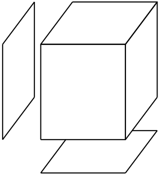

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Faces of Marginal Polytopes

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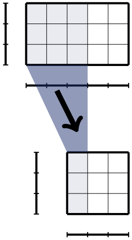
- $r'_i \leqslant r_i$, $1 \leqslant i \leqslant k$.
- projection $\pi : \text{marg}_\Delta(r_1, \ldots, r_k) \longrightarrow \text{marg}_\Delta(r'_1, \ldots, r'_k)$

  by summing parts of marginal directions

  $\Longrightarrow$ facets of $\text{marg}_\Delta(r'_1, \ldots, r'_k)$ [Eriksson et al.]

  lift to faces of $\text{marg}_\Delta(r_1, \ldots, r_k)$



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Faces of Marginal Polytopes

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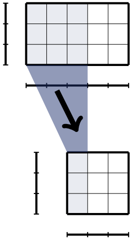

- $r_i' \leqslant r_i$, $1 \leqslant i \leqslant k$.
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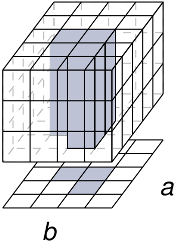

- Face Lemma (for $P(a, b, c)$) [BHNP]
  - F_{ab} fiber over subset of (a, b)-margin, etc.
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    $F_{ab} \cup F_{ac} \cup F_{ac}$

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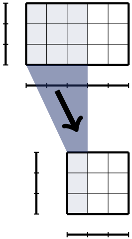

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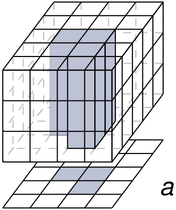
- $P(a, b, c)$ is 3-neighborly, but not 4-neighborly [BHNP,Kahle]



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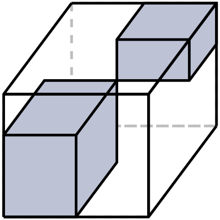
Facets of Marginal Polytopes

Theorem $\Delta := \{\{1,2\},\{1,3\},\{2,3\}\}$ [Eriksson et al.] [BHNP]

$\Longrightarrow$ The number of facets of $\mathrm{marg}_\Delta(a,b,c)$ is at least $\frac{1}{2}(2^a-2)(2^b-2)(2^c-2)+ab+ac+bc$



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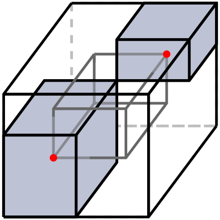

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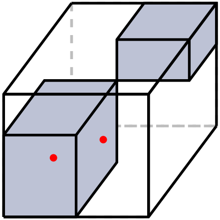

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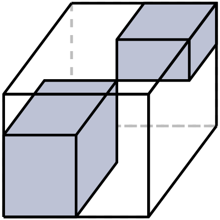

- ▶ more general: Γ graph
  - ▷ valid inequalities given by lift of facet of $\text{marg}_\Gamma(2,\ldots,2)$
  - ▷ correspond to cycle inequalities of $\text{cut}(\Gamma \star s)$
  - ▷ can separate in polynomial time [Barahona, Mahjoub] [Jaakola, Sontag]

Facets of Marginal Polytopes

Theorem [BHNP]

- $\Delta := \{\{1,2\},\{1,3\},\{2,3\}\}$,
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$$\Longrightarrow \quad V := \bigcup_{i=1}^{3} \bigcup_{j=1}^{3} \{(i,i,j),(i,j,i),(j,i,\sigma(i))\}$$

  - is vertex set of a facet of $\mathrm{marg}_\Delta(3,3,3)$
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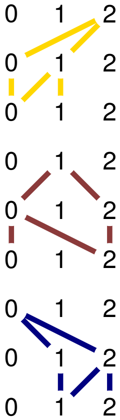

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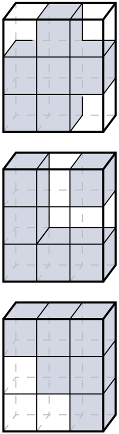

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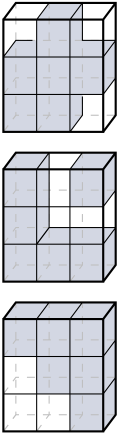
$\longrightarrow$ not all facets come from lifts



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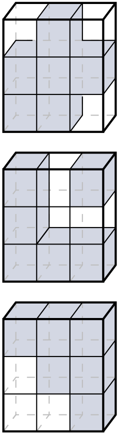

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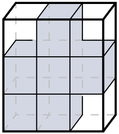

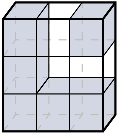

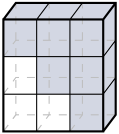

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- separation?