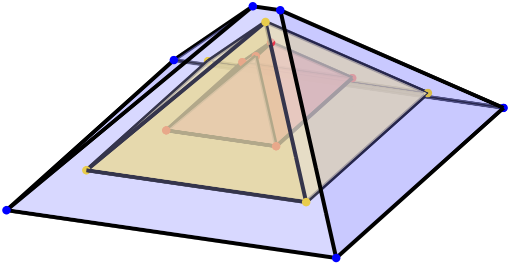
Polyhedral Adjunction Theory

Sydney, November 2012

TECHNISCHE UNIVERSITÄT DARMSTADT



Andreas Paffenholz

joint with

Sandra Di Rocco (Stockholm)

Christian Haase (Frankfurt)

Benjamin Nill (Case Western)

arXiv:1105.2415

Classical Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Polarized variety: pair (X, L), where
  - X is a projective variety of dimension n
  - L is an ample line bundle on X.
  - assume that X is $\mathbb{Q}$-Gorenstein of index r, i.e. $r \cdot K_X$ is Cartier

Classical Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Polarized variety: pair (X, L), where
  - X is a projective variety of dimension n
  - L is an ample line bundle on X.
  - assume that X is $\mathbb{Q}$-Gorenstein of index r, i.e. $r \cdot K_X$ is Cartier
- adjunction theory: study adjoint linear systems $t \cdot L + \quad K_X$

Classical Adjunction Theory

- Polarized variety: pair (X, L), where
  - X is a projective variety of dimension n
  - L is an ample line bundle on X.
  - assume that X is $\mathbb{Q}$-Gorenstein of index r, i.e. $r \cdot K_X$ is Cartier
- adjunction theory: study adjoint linear systems $L + {}^{1}/_{t} \cdot K_X$

Classical Adjunction Theory

- Polarized variety: pair (X, L), where
  - X is a projective variety of dimension n
  - L is an ample line bundle on X.
  - assume that X is $\mathbb{Q}$-Gorenstein of index r, i.e. $r \cdot K_X$ is Cartier
- adjunction theory: study adjoint linear systems $L + {}^1/_t \cdot K_X$
- two invariants
  - nef-value: $\tau := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ ample }))^{-1}$

Classical Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Polarized variety: pair (X, L), where
  - X is a projective variety of dimension n
  - L is an ample line bundle on X.
  - assume that X is $\mathbb{Q}$-Gorenstein of index r, i.e. $r \cdot K_X$ is Cartier
- adjunction theory: study adjoint linear systems $\quad L + {}^1\!/_t \cdot K_X$
- two invariants
  - nef-value: $\tau := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ ample }))^{-1}$
  - unnormalized spectral value: $\mu := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ big }))^{-1}$

Classical Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Polarized variety: pair (X, L), where
  - X is a projective variety of dimension n
  - L is an ample line bundle on X.
  - assume that X is $\mathbb{Q}$-Gorenstein of index r, i.e. $r \cdot K_X$ is Cartier
- adjunction theory: study adjoint linear systems $L + {}^1\!/_t \cdot K_X$
- two invariants
  - nef-value: $\tau := (\sup\,(c \in \mathbb{R} \mid L + c \cdot K_X \text{ ample }))^{-1}$
  - unnormalized spectral value: $\mu := (\sup\,(c \in \mathbb{R} \mid L + c \cdot K_X \text{ big }))^{-1}$

  ||

  (effective threshold)$^{-1}$

Algorithmic Discrete Mathematics polymake

Classical Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Polarized variety: pair (X, L), where
  - X is a projective variety of dimension n
  - L is an ample line bundle on X.
  - assume that X is $\mathbb{Q}$-Gorenstein of index r, i.e. $r \cdot K_X$ is Cartier
- adjunction theory: study adjoint linear systems $L + {}^1/_t \cdot K_X$
- two invariants
  - nef-value: $\tau := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ ample }))^{-1}$
  - unnormalized spectral value: $\mu := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ big }))^{-1}$

$$\|$$

$$(\text{ effective threshold })^{-1} = -(\text{Kodeira energy})$$

Classical Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Polarized variety: pair (X, L), where
  - X is a projective variety of dimension n
  - L is an ample line bundle on X.
  - assume that X is $\mathbb{Q}$-Gorenstein of index r, i.e. $r \cdot K_X$ is Cartier
- adjunction theory: study adjoint linear systems $L + {}^{1}/_{t} \cdot K_X$
- two invariants
  - nef-value: $\tau := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ ample }))^{-1}$
  - unnormalized spectral value: $\mu := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ big }))^{-1}$

||

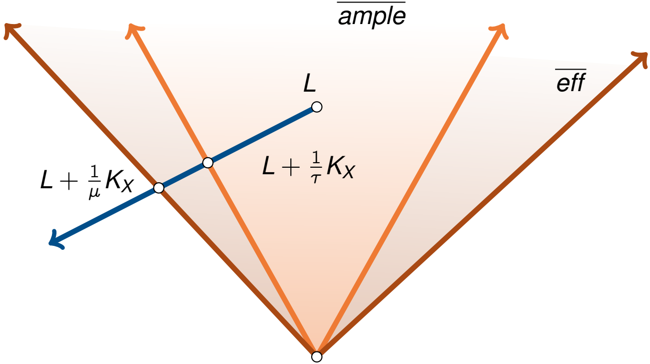
(effective threshold $)^{-1}$ $=$ - (Kodeira energy)

- (ample $\Longrightarrow$ big) $\Longrightarrow$ $\mu \leq \tau$

Polyhedral Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ ample }))^{-1}$
- unnormalized spectral value: $\mu := (\sup(c \in \mathbb{R} \mid L + c \cdot K_X \text{ big }))^{-1}$



Spectral Value and Nef-Value

TECHNISCHE UNIVERSITÄT DARMSTADT

- general bounds for μ and τ
  - $\tau, \mu \in \mathbb{Q}$
  - $r \cdot K_X$ Cartier $\implies \tau \leq r(n+1)$. [Kawamata's Rationality Theorem]

Algorithmic Discrete Mathematics polymake

Spectral Value and Nef-Value

TECHNISCHE UNIVERSITÄT DARMSTADT

- general bounds for μ and τ
  - $\tau, \mu \in \mathbb{Q}$
  - $r \cdot K_X$ Cartier $\implies \tau \leq r(n+1)$. [Kawamata's Rationality Theorem]
  - $\mu \leq n+1$ [Beltrametti, Sommese '95]

    (equality only for $(X, L) = (\mathbb{P}, \mathcal{O}(1))$)

Spectral Value and Nef-Value

TECHNISCHE UNIVERSITÄT DARMSTADT

- general bounds for μ and τ
  - $\tau, \mu \in \mathbb{Q}$
  - $r \cdot K_X$ Cartier $\implies \tau \leq r(n+1)$. [Kawamata's Rationality Theorem]
  - $\mu \leq n+1$ [Beltrametti, Sommese '95]

    (equality only for $(X, L) = (\mathbb{P}, \mathcal{O}(1))$
- bounds, if X is non-singular
  - $\tau \leq n+1$, with equality only if $(X, L) = (\mathbb{P}^n, O(1))$

Spectral Value and Nef-Value

TECHNISCHE UNIVERSITÄT DARMSTADT

- general bounds for μ and τ
  - $\tau, \mu \in \mathbb{Q}$
  - $r \cdot K_X$ Cartier $\implies \tau \leq r(n+1)$. [Kawamata's Rationality Theorem]
  - $\mu \leq n+1$ [Beltrametti, Sommese '95]

    (equality only for $(X, L) = (\mathbb{P}, \mathcal{O}(1))$)
- bounds, if X is non-singular
  - $\tau \leq n+1$, with equality only if $(X, L) = (\mathbb{P}^n, O(1))$
  - Classification for $\tau > n-3$ [Fujita, Beltrametti/Sommese, ...]

Spectral Value and Nef-Value

TECHNISCHE UNIVERSITÄT DARMSTADT

- general bounds for μ and τ
  - $\tau, \mu \in \mathbb{Q}$
  - $r \cdot K_X$ Cartier $\implies \tau \leq r(n+1)$. [Kawamata's Rationality Theorem]
  - $\mu \leq n+1$ [Beltrametti, Sommese '95]

    (equality only for $(X, L) = (\mathbb{P}, \mathcal{O}(1))$)

- bounds, if X is non-singular
  - $\tau \leq n+1$, with equality only if $(X, L) = (\mathbb{P}^n, O(1))$
  - Classification for $\tau > n-3$ [Fujita, Beltrametti/Sommese, ...]

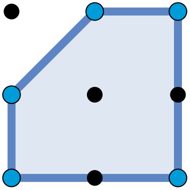
- $\mathbb{Q}$-normality conjecture for polarized manifolds:

$$\mu > \frac{n+1}{2} \quad \implies \quad \mu = \tau \quad (:\iff \quad X\ \mathbb{Q}\text{-normal})$$

Lattice Polytopes

TECHNISCHE
UNIVERSITÄT
DARMSTADT

- lattice polytope : $P := \operatorname{conv}(v_1, \dots, v_k)$ for $v_j \in \mathbb{Z}^n$.

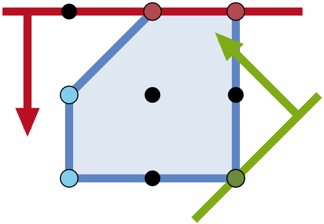


Lattice Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

- lattice polytope : $P := \operatorname{conv}(v_1, \dots, v_k)$ for $v_j \in \mathbb{Z}^n$.

  $P = \{ x \mid \mathbf{a}_i^t x \le b_i \quad 1 \le i \le m \}$ for $\mathbf{a}_i, b_i$ integral



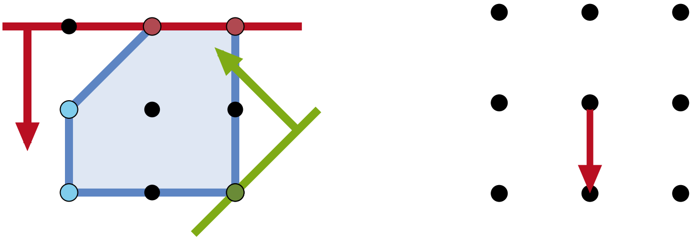
Lattice Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

- **lattice polytope**: $P := \mathrm{conv}(v_1, \dots, v_k)$ for $v_j \in \mathbb{Z}^n$.

  $P = \{ x \mid \mathbf{a}_i^t x \leq b_i \quad 1 \leq i \leq m \}$ for $\mathbf{a}_i, b_i$ integral

  - normal cone at a face F: cone of all linear functionals that take their minimum at F



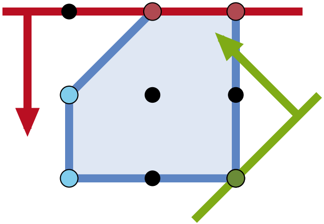
Lattice Polytopes

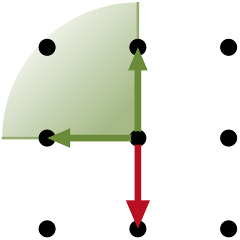
TECHNISCHE UNIVERSITÄT DARMSTADT

- lattice polytope : $P := \operatorname{conv}(v_1, \dots, v_k)$ for $v_j \in \mathbb{Z}^n$.

  $P = \{ x \mid \mathbf{a}_i^t x \le b_i \quad 1 \le i \le m \}$ for $\mathbf{a}_i, b_i$ integral

- normal cone at a face F: cone of all linear functionals that take their minimum at F





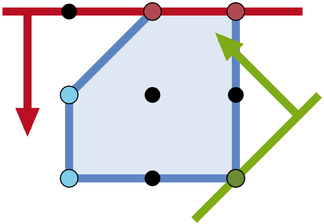
Lattice Polytopes

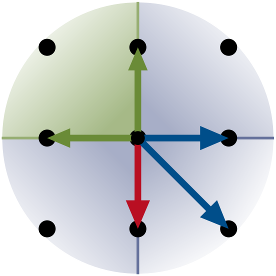
TECHNISCHE
UNIVERSITÄT
DARMSTADT

▶ lattice polytope : $P := \mathrm{conv}(v_1, \dots, v_k)$ for $v_j \in \mathbb{Z}^n$.

$P = \{ x \mid \mathbf{a}_i^t x \le b_i \quad 1 \le i \le m \}$ for $\mathbf{a}_i, b_i$ integral

▷ normal cone at a face F: cone of all linear functionals that take their minimum at F



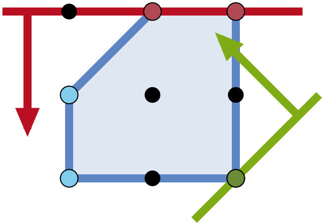


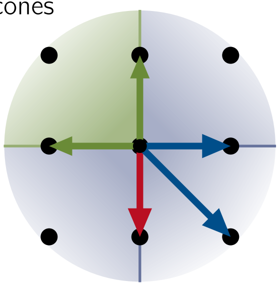
Lattice Polytopes

- **lattice polytope:** $P := \operatorname{conv}(v_1, \ldots, v_k)$ for $v_j \in \mathbb{Z}^n$.

  $P = \{ x \mid \mathbf{a}_i^t x \le b_i \quad 1 \le i \le m \}$ for $\mathbf{a}_i, b_i$ integral

- normal cone at a face F: cone of all linear functionals that take their minimum at F
- normal fan $\mathrm{NF}(P)$: collection of all normal cones





Polyhedral Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

▷ $P \subseteq \mathbb{R}^n$ lattice polytope, $\dim P = n$

$$P = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \mathbf{a}_i^t \mathbf{x} \le b_i\,, \quad 1 \le i \le m \right\}, \qquad \mathbf{a}_i \in \mathbb{Z}^n \text{ primitive, irredundant, } b_i \in \mathbb{Z}$$

Polyhedral Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- $P \subseteq \mathbb{R}^n$ lattice polytope, $\dim P = n$

$$P = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \mathbf{a}_i^t \mathbf{x} \le b_i\,, \quad 1 \le i \le m \right\}, \qquad \mathbf{a}_i \in \mathbb{Z}^n \text{ primitive, irredundant,} \quad b_i \in \mathbb{Z}$$

- $c \geq 0$
- adjoint polytope $P^{(c)}$:

  points with lattice distance at least c from each facet.

Polyhedral Adjunction Theory

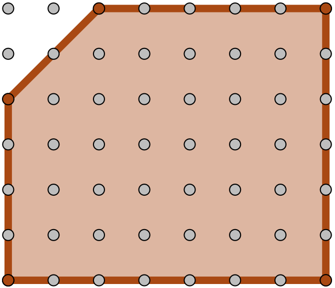
TECHNISCHE UNIVERSITÄT DARMSTADT

- $P \subseteq \mathbb{R}^n$ lattice polytope, $\dim P = n$

$$P = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \mathbf{a}_i^t \mathbf{x} \le b_i\,, \quad 1 \le i \le m \right\}, \qquad \mathbf{a}_i \in \mathbb{Z}^n \text{ primitive, irredundant, } b_i \in \mathbb{Z}$$

- $c \ge 0$
- adjoint polytope $P^{(c)}$:

  points with lattice distance at least c from each facet.



$c = 0$

Algorithmic Discrete Mathematics polymake

Polyhedral Adjunction Theory

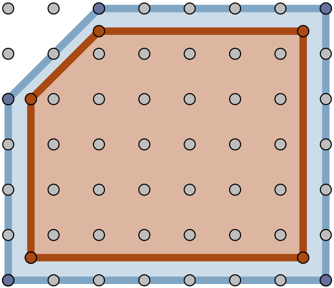
- $P \subseteq \mathbb{R}^n$ lattice polytope, $\dim P = n$

$$P = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \mathbf{a}_i^t \mathbf{x} \leq b_i \,, \quad 1 \leq i \leq m \right\},$$

$\mathbf{a}_i \in \mathbb{Z}^n$ primitive, irredundant, $b_i \in \mathbb{Z}$

- $c \geq 0$
- adjoint polytope $P^{(c)}$:

  points with lattice distance at least c from each facet.



$c = \frac{1}{2}$

Polyhedral Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

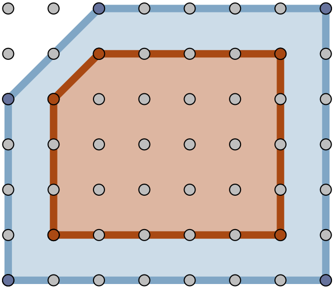
- ▷ $P \subseteq \mathbb{R}^n$ lattice polytope, $\dim P = n$

$$P = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \mathbf{a}_i^t \mathbf{x} \le b_i\,, \quad 1 \le i \le m \right\},$$

$\mathbf{a}_i \in \mathbb{Z}^n$ primitive, irredundant, $b_i \in \mathbb{Z}$

- ▷ $c \geq 0$
- ▶ adjoint polytope $P^{(c)}$:

  points with lattice distance at least c from each facet.



$c = 1$

Algorithmic Discrete Mathematics polymake

Polyhedral Adjunction Theory

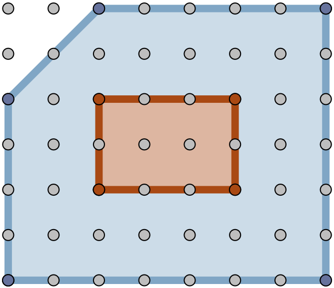
- $P \subseteq \mathbb{R}^n$ lattice polytope, $\dim P = n$

$$P = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \mathbf{a}_i^t \mathbf{x} \leq b_i\,, \quad 1 \leq i \leq m \right\},$$

$\mathbf{a}_i \in \mathbb{Z}^n$ primitive, irredundant, $b_i \in \mathbb{Z}$

- $c \geq 0$
- adjoint polytope $P^{(c)}$:

  points with lattice distance at least c from each facet.



$c = 2$

Polyhedral Adjunction Theory

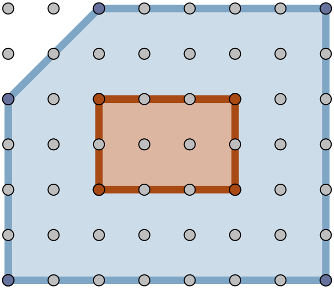
TECHNISCHE UNIVERSITÄT DARMSTADT

- $P \subseteq \mathbb{R}^n$ lattice polytope, $\dim P = n$

  $P = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \mathbf{a}_i^t \mathbf{x} \leq b_i\,, \quad 1 \leq i \leq m \right\}$, $\mathbf{a}_i \in \mathbb{Z}^n$ primitive, irredundant, $b_i \in \mathbb{Z}$

- $c \geq 0$
- adjoint polytope $P^{(c)}$:

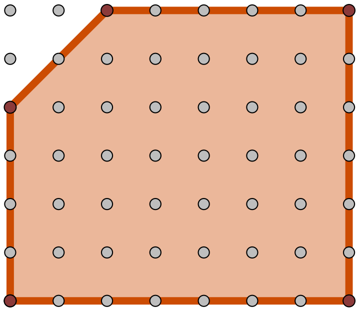
  points with lattice distance at least c from each facet.

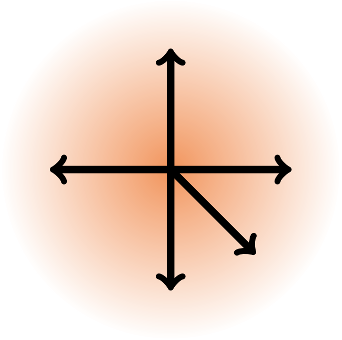


$$P^{(c)} \quad = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \mathbf{a}_i^t \mathbf{x} \leq b_i - c\,,\; 1 \leq i \leq m \right\}$$

$c = 2$

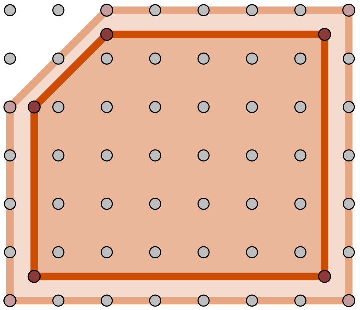
Invariants for Lattice Polytopes

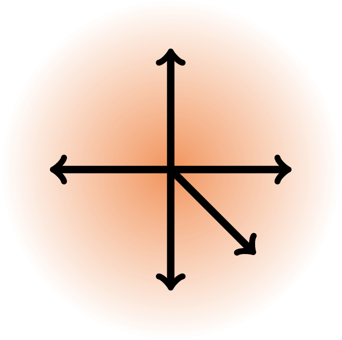




- $c = 0$

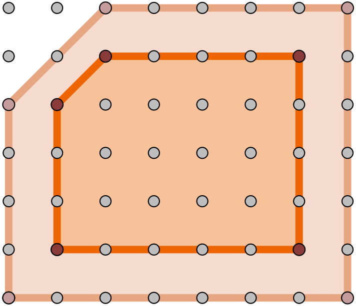
Invariants for Lattice Polytopes

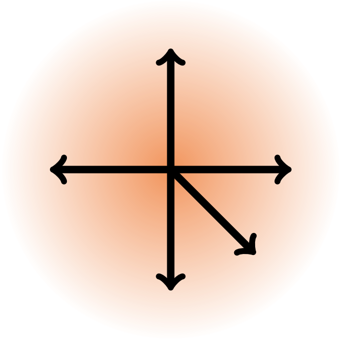




- $c = \frac{1}{2}$

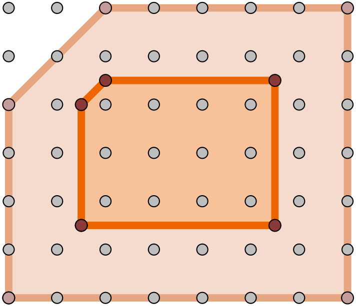
Invariants for Lattice Polytopes

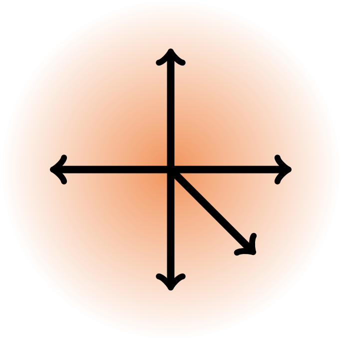




- $c = 1$

Invariants for Lattice Polytopes

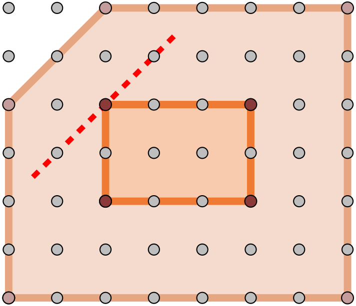


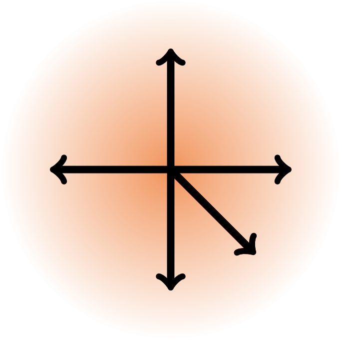


- $c = \frac{3}{2}$

Invariants for Lattice Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT



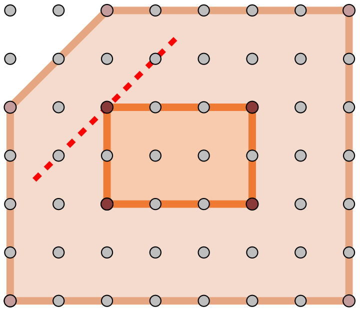


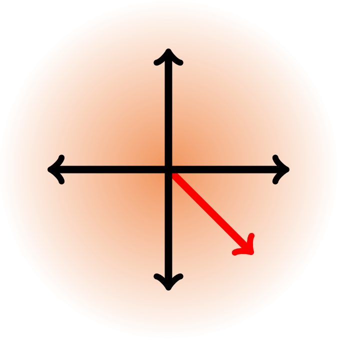
- $c = 2$

Invariants for Lattice Polytopes

TECHNISCHE
UNIVERSITÄT
DARMSTADT

- nef-value: $\tau := \inf \left(c > 0 \mid P \text{ and } P^{(c)} \text{ are combinatorially different } \right)^{-1}$

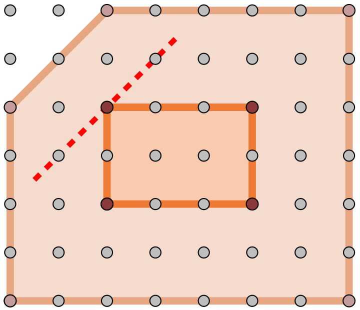


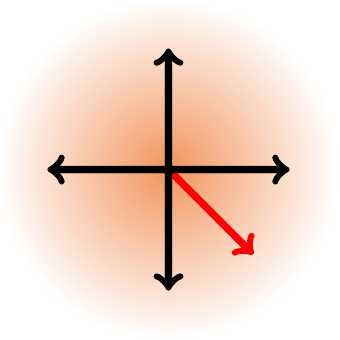


- $c = 2$
- $\tau = \frac{1}{2}$

Invariants for Lattice Polytopes

- nef-value: $\tau := \sup(\, c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)})\,)^{-1}$

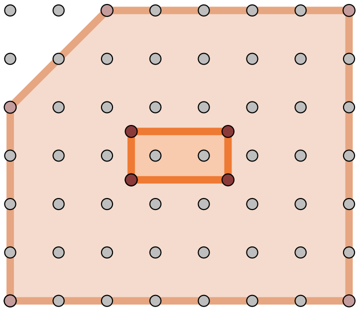


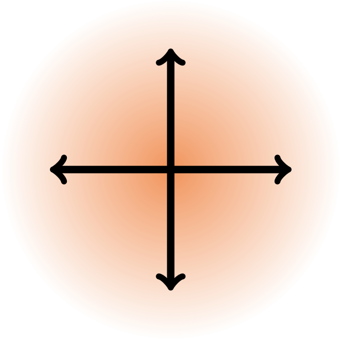


- $c = 2$
- $\tau = \frac{1}{2}$

Invariants for Lattice Polytopes

- nef-value: $\tau := \sup\left(c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)}) \right)^{-1}$

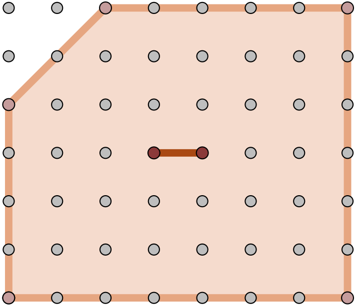


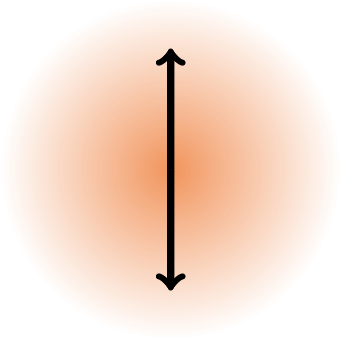


- $c = \frac{5}{2}$
- $\tau = \frac{1}{2}$

Invariants for Lattice Polytopes

- nef-value: $\tau := \sup(\, c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)})\,)^{-1}$



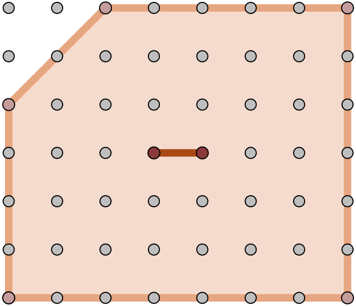


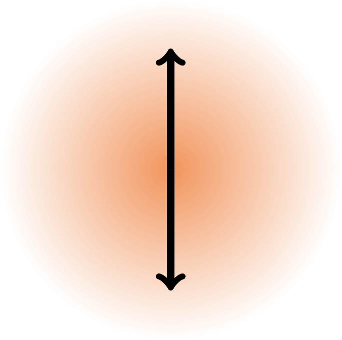
- $c = 3$
- $\tau = \frac{1}{2}$

Invariants for Lattice Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)}))^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)} \neq \varnothing)^{-1}$



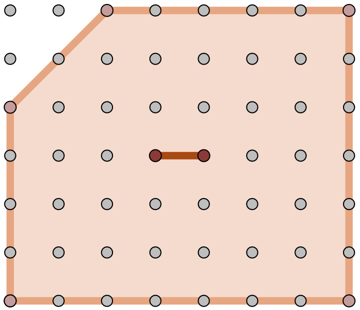


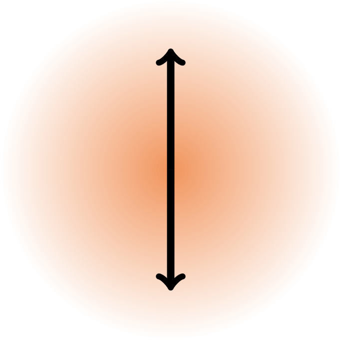
- $c = 3$
- $\tau = \frac{1}{2}$
- $\mu = \frac{1}{3}$

Invariants for Lattice Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)}))^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)}$ is full-dimensional $)^{-1}$

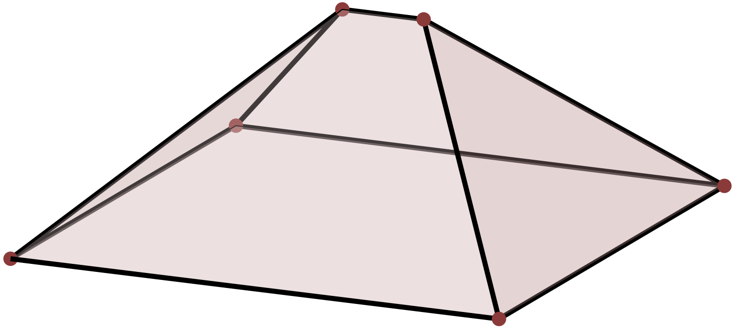




- $c = 3$
- $\tau = \frac{1}{2}$
- $\mu = \frac{1}{3}$

Invariants for Lattice Polytopes

- nef-value: $\tau := \sup(c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)}))^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)}$ is full-dimensional $)^{-1}$

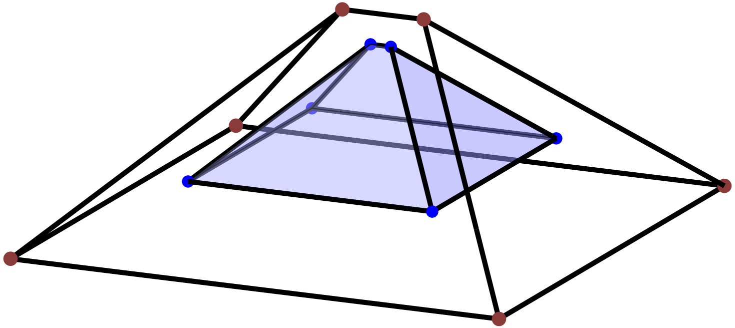


- $c = 0$

Invariants for Lattice Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(\, c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)})\,)^{-1}$
- spectral value: $\mu := \sup(\, c \geq 0 \mid P^{(c)}$ is full-dimensional $)^{-1}$



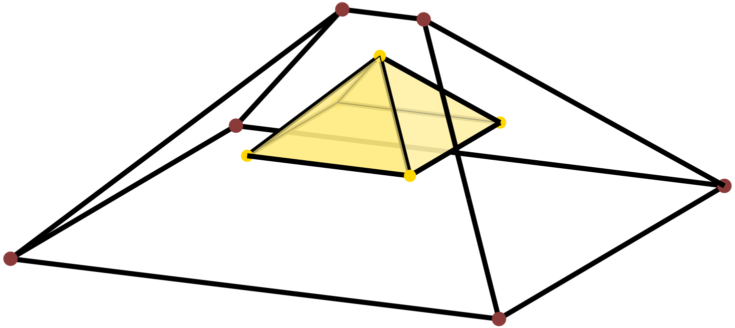
- $c = \frac{3}{2}$

Algorithmic Discrete Mathematics polymake

Invariants for Lattice Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)}))^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)}$ is full-dimensional $)^{-1}$

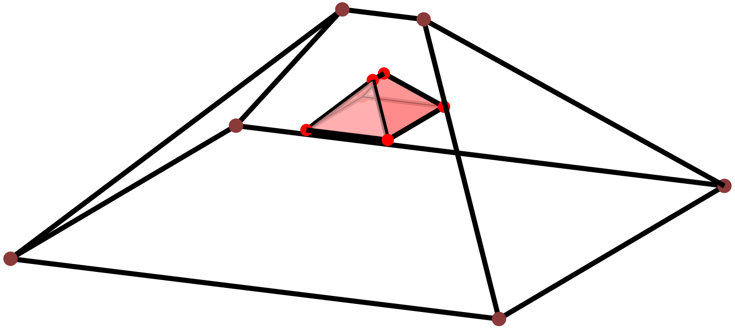


- $c = 2$
- $\tau = \frac{1}{2}$

Algorithmic Discrete Mathematics polymake

Invariants for Lattice Polytopes

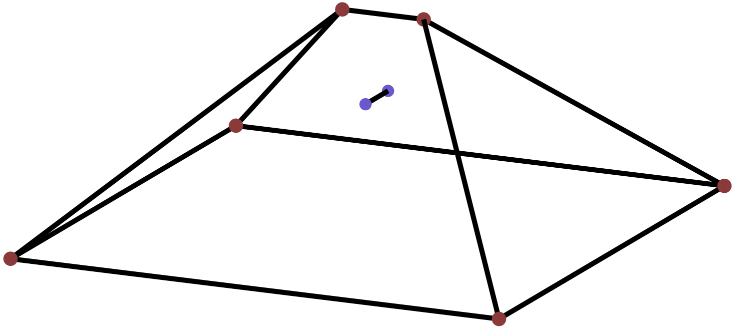
- nef-value: $\tau := \text{sup}(\, c > 0 \mid \text{NF}(P) = \text{NF}(P^{(c)})\,)^{-1}$
- spectral value: $\mu := \text{sup}(\, c \geq 0 \mid P^{(c)} \text{ is full-dimensional}\,)^{-1}$



- $c = \frac{5}{2}$
- $\tau = \frac{1}{2}$

Invariants for Lattice Polytopes

- nef-value: $\tau := \sup(c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)}))^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)}$ is full-dimensional $)^{-1}$



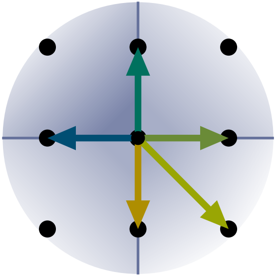
- $c = 3$
- $\tau = \frac{1}{2}$
- $\mu = \frac{1}{3}$

Lattice Polytopes and Toric Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

complete polyhedral fan $\Sigma_X \subseteq \mathbb{R}^n$ $\longleftrightarrow$ projective toric variety X_Σ

ray generators $u_i \in \mathbb{Z}^n$ $\longleftrightarrow$ torus invariant prime divisors D_i



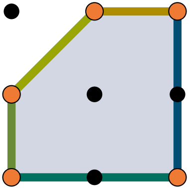
Lattice Polytopes and Toric Varieties

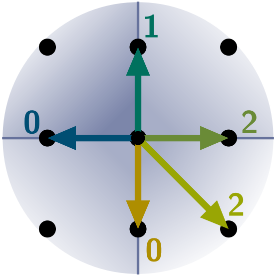
complete polyhedral fan $\Sigma_X \subseteq \mathbb{R}^n$ $\longleftrightarrow$ projective toric variety X_Σ

ray generators $u_i \in \mathbb{Z}^n$ $\longleftrightarrow$ torus invariant prime divisors D_i

lattice polytope $\longleftrightarrow$ polarized toric variety

$P := \{x \in \mathbb{R}^d \mid \langle v, u_i \rangle \geq a_i\}$ $\qquad$ $(X_P, \mathcal{O}_X(\sum -a_i \cdot D_i))$





Correspondence to Toric Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)}))^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)}$ is full-dimensional $)^{-1}$

Correspondence to Toric Varieties

- nef-value: $\tau := \sup(c > 0 \mid \mathrm{NF}(P) = \mathrm{NF}(P^{(c)}))^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)} \text{ is full-dimensional})^{-1}$

- (X, L) polarized toric variety
  - $\mathbb{Q}$-Gorenstein
  - L ample
  - Σ associated fan with rays

divisor D on X is ample $\longleftrightarrow$ $\mathrm{NF}(P_D) = \Sigma$

Correspondence to Toric Varieties

- nef-value: $\tau := \sup(c > 0 \mid L + c \cdot K_X \text{ is ample })^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)} \text{ is full-dimensional})^{-1}$

- (X, L) polarized toric variety
  - $\mathbb{Q}$-Gorenstein
  - L ample
  - Σ associated fan with rays

divisor D on X is ample $\longleftrightarrow$ $\mathrm{NF}(P_D) = \Sigma$

Correspondence to Toric Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(c > 0 \mid L + c \cdot K_X \text{ is ample })^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid P^{(c)} \text{ is full-dimensional})^{-1}$

▷ (X, L) polarized toric variety

▷ $\mathbb{Q}$-Gorenstein ▷ L ample ▷ Σ associated fan with rays

divisor D on X is ample $\longleftrightarrow$ $\mathrm{NF}(P_D) = \Sigma$

divisor D on X is big $\longleftrightarrow$ $\dim P_D = \dim X$

Correspondence to Toric Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(c > 0 \mid L + c \cdot K_X \text{ is ample })^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid L + c \cdot K_X \text{ big })^{-1}$

- (X, L) polarized toric variety
  - $\mathbb{Q}$-Gorenstein
  - L ample
  - Σ associated fan with rays

$$\text{divisor } D \text{ on } X \text{ is ample} \quad \longleftrightarrow \quad \mathrm{NF}(P_D) = \Sigma$$

$$\text{divisor } D \text{ on } X \text{ is big} \quad \longleftrightarrow \quad \dim P_D = \dim X$$

Correspondence to Toric Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- nef-value: $\tau := \sup(c > 0 \mid L + c \cdot K_X \text{ is ample })^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid L + c \cdot K_X \text{ big })^{-1}$

- (X, L) polarized toric variety
  - $\mathbb{Q}$-Gorenstein
  - L ample
  - Σ associated fan with rays

divisor D on X is ample $\longleftrightarrow$ $\mathrm{NF}(P_D) = \Sigma$

divisor D on X is big $\longleftrightarrow$ $\dim P_D = \dim X$

X $\mathbb{Q}$-Gorenstein $\longleftrightarrow$ K_X $\mathbb{Q}$-Cartier $\iff$ $\tau > 0$

Correspondence to Toric Varieties

- nef-value: $\tau := \sup(c > 0 \mid L + c \cdot K_X \text{ is ample })^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid L + c \cdot K_X \text{ big })^{-1}$

▷ (X, L) polarized toric variety

▷ $\mathbb{Q}$-Gorenstein ▷ L ample ▷ Σ associated fan with rays

$$\begin{aligned} \text{divisor } D \text{ on } X \text{ is ample} &\longleftrightarrow \text{NF}(P_D) = \Sigma \\ \text{divisor } D \text{ on } X \text{ is big} &\longleftrightarrow \dim P_D = \dim X \\ X\ \mathbb{Q}\text{-Gorenstein} &\longleftrightarrow K_X\ \mathbb{Q}\text{-Cartier} \iff \tau > 0 \end{aligned}$$

- let $0^{-1} = \infty$: $\implies$ polyhedral definition does not need $\mathbb{Q}$-Gorenstein

Correspondence to Toric Varieties

- nef-value: $\tau := \sup(c > 0 \mid L + c \cdot K_X$ is ample $)^{-1}$
- spectral value: $\mu := \sup(c \geq 0 \mid L + c \cdot K_X$ big $)^{-1}$

- (X, L) polarized toric variety
  - $\mathbb{Q}$-Gorenstein
  - L ample
  - Σ associated fan with rays

$$\text{divisor } D \text{ on } X \text{ is ample} \longleftrightarrow \mathrm{NF}(P_D) = \Sigma$$

$$\text{divisor } D \text{ on } X \text{ is big} \longleftrightarrow \dim P_D = \dim X$$

$$X\ \mathbb{Q}\text{-Gorenstein} \longleftrightarrow K_X\ \mathbb{Q}\text{-Cartier} \iff \tau > 0$$

- let $0^{-1} = \infty$: $\implies$ polyhedral definition does not need $\mathbb{Q}$-Gorenstein

Polyhedral Adjunction Theory $\supsetneq$ **classical Adjunction Theory**

Computational Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- combinatorial definition allows efficient computations

Computational Adjunction Theory

- combinatorial definition allows efficient computations
- `polymake`: computations in polyhedral geometry and related fields

Computational Adjunction Theory

- combinatorial definition allows efficient computations
- `polymake`: computations in polyhedral geometry and related fields
  - lattice polytopes, toric geometry, tropical geometry, ...
  - fully programmable interface: `perl`, C++
  - symmetric polyhedra, sage integration, GAP/Singular interface
- http://polymake.org

Computational Adjunction Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- combinatorial definition allows efficient computations
- polymake: computations in polyhedral geometry and related fields
  - lattice polytopes, toric geometry, tropical geometry, ...
  - fully programmable interface: perl, C++
  - symmetric polyhedra, sage integration, GAP/Singular interface
- http://polymake.org
- polymake extension

  PolyhedralAdjunction

  https://github.com/apaffenholz/polyhedral_adjunction.git

  ToricVarieties

  https://github.com/lkastner/polymake_toric.git

Large spectral value implies Cayley

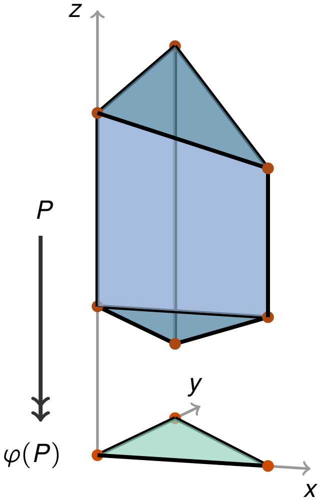
- ▷ P n-dimensional lattice polytope
- ▶ P is a Cayley sum

$$:\iff$$

P has a projection onto standard simplex

$$\varphi : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^m \quad \text{lattice projection}$$

such that $\quad \varphi(P) = \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_m).$



Large spectral value implies Cayley

- ▷ P n-dimensional lattice polytope
- ▶ P is a Cayley sum

$$:\Longleftrightarrow$$

P has a projection onto standard simplex

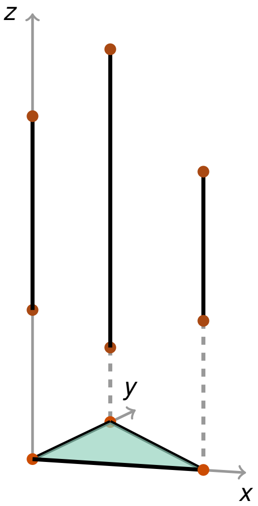
$$\varphi : \mathbb{R}^n \longrightarrow\!\!\!\!\rightarrow \mathbb{R}^m \quad \text{lattice projection}$$

such that $\quad \varphi(P) = \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_m)$.

- ▷ equivalently

$$P \;=\; \operatorname{conv}(P_0 \times \{\mathbf{0}\}, P_1 \times \{\mathbf{e}_1\}, \dots, P_m \times \{\mathbf{e}_m\})$$

for lattice polytopes $P_0, \dots, P_m \in \mathbb{R}^{n-m}$.



Large spectral value implies Cayley

- P n-dimensional lattice polytope
- P is a Cayley sum

  $:\Longleftrightarrow$

  P has a projection onto standard simplex

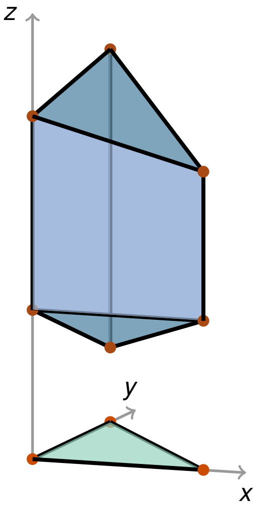
  $$\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ lattice projection}$$

  such that $\varphi(P) = \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_m)$.

- equivalently

  $$P = \operatorname{conv}(P_0 \times \{\mathbf{0}\}, P_1 \times \{\mathbf{e}_1\}, \dots, P_m \times \{\mathbf{e}_m\})$$

  for lattice polytopes $P_0, \dots, P_m \in \mathbb{R}^{n-m}$.



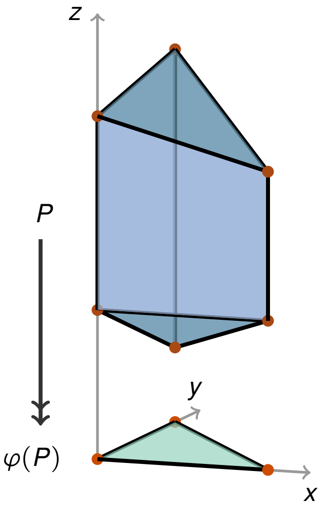
Large spectral value implies Cayley

TECHNISCHE UNIVERSITÄT DARMSTADT

- ▷ P n-dimensional lattice polytope
- ▶ Theorem

$$\mu \geq \frac{n+2}{2} \implies P \text{ is a Cayley sum of lattice polytopes in } R^m$$

$$m \leq \lfloor 2(n+1-\mu) \rfloor$$



Large spectral value implies Cayley

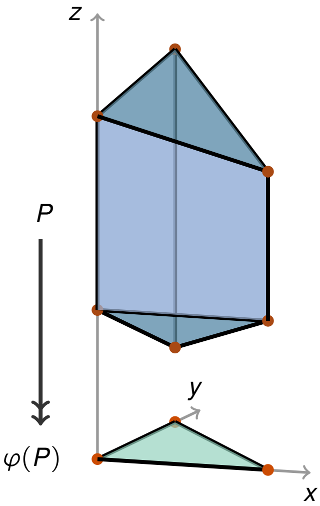
TECHNISCHE UNIVERSITÄT DARMSTADT

- P n-dimensional lattice polytope
- Theorem

  $\mu \geq \frac{n+2}{2} \implies P$ is a Cayley sum of lattice polytopes in R^m

  $m \leq \lfloor 2(n+1-\mu) \rfloor$

  - $\mu \geq \frac{n+2}{2} \iff n > m$



Large spectral value implies Cayley

TECHNISCHE UNIVERSITÄT DARMSTADT

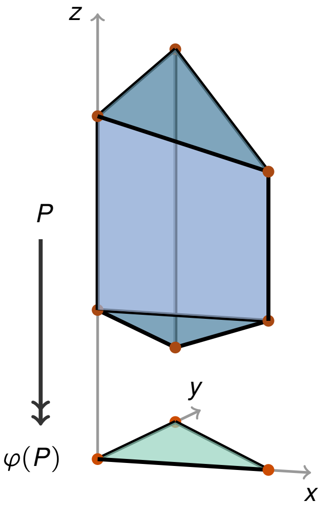
- P n-dimensional lattice polytope
- **Theorem**

  $\mu \geq \frac{n+2}{2} \implies P$ is a Cayley sum of lattice polytopes in R^m

  $m \leq \lfloor 2(n+1-\mu) \rfloor$

  - $\mu \geq \frac{n+2}{2} \iff n > m$
- **Corollary**

  $\mu \geq \frac{n+2}{2} \implies P$ has lattice width one



Large spectral value implies Cayley

TECHNISCHE UNIVERSITÄT DARMSTADT

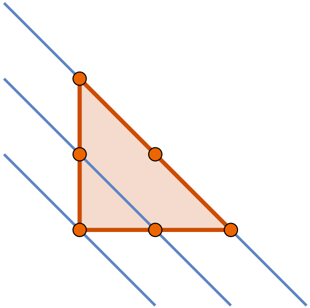
- P n-dimensional lattice polytope

- **Theorem**

  $\mu \geq \frac{n+2}{2} \implies P$ is a Cayley sum of lattice polytopes in R^m

  $$m \leq \lfloor 2(n+1-\mu) \rfloor$$

  - $\mu \geq \frac{n+2}{2} \iff n > m$



- **Corollary**

  $\mu \geq \frac{n+2}{2} \implies P$ has lattice width one

- best possible bound:

  $$X = (\mathbb{P}^2, O(2)) \rightsquigarrow P = 2\Delta_2, \ \mu = \frac{n+1}{2}$$

Large spectral value implies Cayley

TECHNISCHE UNIVERSITÄT DARMSTADT

- ▷ (X, L) polarized toric variety in dimension n
- ▶ Theorem

  $$\mu \geq \frac{n+2}{2} \implies \text{morphism } \pi : \mathbb{P}(H_0 \oplus H_1 \oplus \cdots \oplus H_m) \longrightarrow X$$

  H_i: line bundles

  on toric variety in dimension $\leq 2(n+1-\mu)$

  - ▷ $\mu \geq \frac{n+2}{2} \iff n > m$
- ▶ Corollary

  $$\mu \geq \frac{n+2}{2} \implies P \text{ has lattice width one}$$

- ▷ best possible bound:

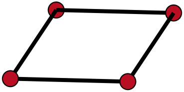
  $$X = (\mathbb{P}^2, O(2)) \rightsquigarrow P = 2\Delta_2\,,\ \mu = \frac{n+1}{2}$$

Proof Sketch

- Theorem $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m

Proof Sketch

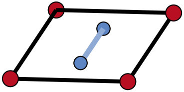
- Theorem $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m
  - let $\mu \geq \frac{n+2}{2}, \quad P = \left\{\mathbf{x} \mid \mathbf{a}_i^t\mathbf{x} \leq b_i\right\}$



Proof Sketch

TECHNISCHE UNIVERSITÄT DARMSTADT

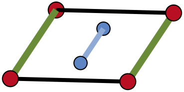
- **Theorem** $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m
  - let $\mu \geq \frac{n+2}{2}, \quad P = \{\mathbf{x} \mid \mathbf{a}_i^t\mathbf{x} \leq b_i\}$
  - core of P: $\operatorname{core}(P) := P^{(\frac{1}{\mu})}$



Proof Sketch

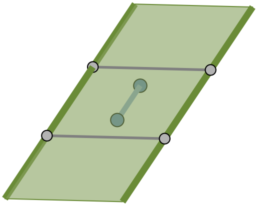
TECHNISCHE UNIVERSITÄT DARMSTADT

- Theorem $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m
  - let $\mu \geq \frac{n+2}{2}, \quad P = \{\mathbf{x} \mid \mathbf{a}_i^t \mathbf{x} \leq b_i\}$
  - core of P: $\operatorname{core}(P) := P^{(\frac{1}{\mu})}$
  - core facet: $\mathbf{a}_i^t\mathbf{x} + \mu \leq b_i$ supporting $\operatorname{aff}\operatorname{core}(P)$



Proof Sketch

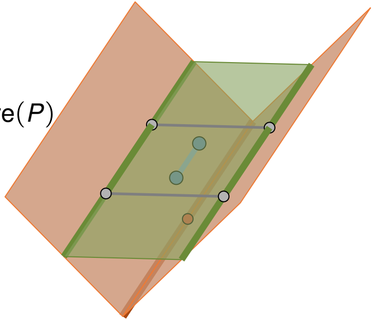
- Theorem $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m
  - let $\mu \geq \frac{n+2}{2}, \quad P = \{\mathbf{x} \mid \mathbf{a}_i^t\mathbf{x} \leq b_i\}$
  - core of P: $\operatorname{core}(P) := P^{(\frac{1}{\mu})}$
  - core facet: $\mathbf{a}_i^t\mathbf{x} + \mu \leq b_i$ supporting $\operatorname{aff}\operatorname{core}(P)$
  - Q: polyhedron spanned by core facets



Proof Sketch

TECHNISCHE UNIVERSITÄT DARMSTADT

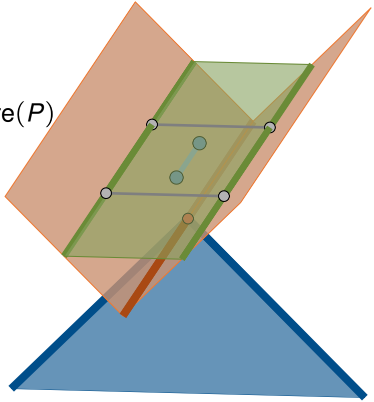
- Theorem $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m
  - let $\mu \geq \frac{n+2}{2}, \quad P = \{\mathbf{x} \mid \mathbf{a}_i^t\mathbf{x} \leq b_i\}$
  - core of P: $\operatorname{core}(P) := P^{(\frac{1}{\mu})}$
  - core facet: $\mathbf{a}_i^t\mathbf{x} + \mu \leq b_i$ supporting $\operatorname{aff}\operatorname{core}(P)$
  - Q: polyhedron spanned by core facets
  - $\sigma := Q \times \{1\}$



Proof Sketch

TECHNISCHE UNIVERSITÄT DARMSTADT

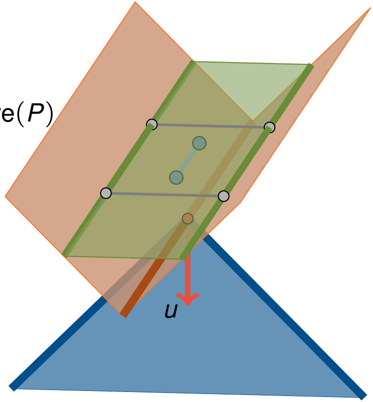
- Theorem $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m
  - let $\mu \geq \frac{n+2}{2}$, $\quad P = \{\mathbf{x} \mid \mathbf{a}_i^t\mathbf{x} \leq b_i\}$
  - core of P: $\mathrm{core}(P) := P^{(\frac{1}{\mu})}$
  - core facet: $\mathbf{a}_i^t\mathbf{x} + \mu \leq b_i$ supporting $\mathrm{aff}\,\mathrm{core}(P)$
  - Q: polyhedron spanned by core facets
  - $\sigma := Q \times \{1\}$
  - $\sigma^\vee$: cone dual to $Q \times \{1\}$



Proof Sketch

TECHNISCHE UNIVERSITÄT DARMSTADT

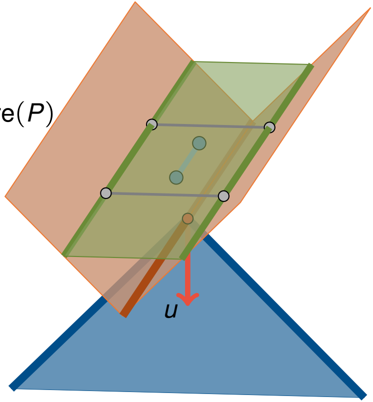
- Theorem $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m
  - let $\mu \geq \frac{n+2}{2}$, $\quad P = \{\mathbf{x} \mid \mathbf{a}_i^t \mathbf{x} \leq b_i\}$
  - core of P: $\text{core}(P) := P^{(\frac{1}{\mu})}$
  - core facet: $\mathbf{a}_i^t \mathbf{x} + \mu \leq b_i$ supporting $\text{aff}\,\text{core}(P)$
  - Q: polyhedron spanned by core facets
  - $\sigma := Q \times \{1\}$
  - $\sigma^\vee$: cone dual to $Q \times \{1\}$
  - contains the functional $\mathbf{u} = (\mathbf{0}, 1)$.



Proof Sketch

TECHNISCHE UNIVERSITÄT DARMSTADT

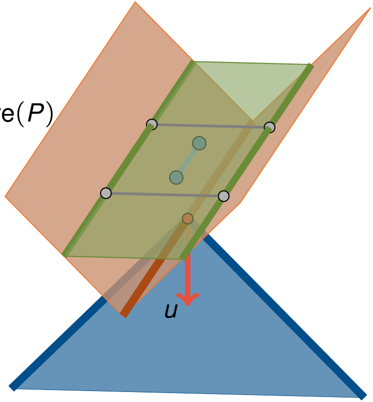
- Theorem $\mu \geq \frac{n+2}{2} \quad \Rightarrow \quad P$ is a Cayley sum of lattice polytopes in R^m
  - let $\mu \geq \frac{n+2}{2}$, $\quad P = \{\mathbf{x} \mid \mathbf{a}_i^t\mathbf{x} \leq b_i\}$
  - core of P: $\operatorname{core}(P) := P^{(\frac{1}{\mu})}$
  - core facet: $\mathbf{a}_i^t\mathbf{x} + \mu \leq b_i$ supporting $\operatorname{aff}\operatorname{core}(P)$
  - Q: polyhedron spanned by core facets
  - $\sigma := Q \times \{1\}$
  - $\sigma^\vee$: cone dual to $Q \times \{1\}$
  - contains the functional $\mathbf{u} = (\mathbf{0}, 1)$.
  - show that $\mathbf{u}$ is sum of lattice points in σ



Proof Sketch

TECHNISCHE UNIVERSITÄT DARMSTADT

- Theorem $\mu \geq \frac{n+2}{2} \Rightarrow P$ is a Cayley sum of lattice polytopes in R^m
  - let $\mu \geq \frac{n+2}{2}$, $P = \{\mathbf{x} \mid \mathbf{a}_i^t \mathbf{x} \leq b_i\}$
  - core of P: $\operatorname{core}(P) := P^{(\frac{1}{\mu})}$
  - core facet: $\mathbf{a}_i^t \mathbf{x} + \mu \leq b_i$ supporting $\operatorname{aff}\operatorname{core}(P)$
  - Q: polyhedron spanned by core facets
  - $\sigma := Q \times \{1\}$
  - $\sigma^\vee$: cone dual to $Q \times \{1\}$
  - contains the functional $\mathbf{u} = (\mathbf{0}, 1)$.
  - show that $\mathbf{u}$ is sum of lattice points in σ
  - this is equivalent to P being Cayley



[Batyrev, Nill]

Algorithmic Discrete Mathematics polymake

Relation to Ehrhart Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Ehrhart Theory:
  - Count lattice points in a polytope P: $|P \cap \mathbb{Z}^n|$.

Algorithmic Discrete Mathematics polymake

Relation to Ehrhart Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Ehrhart Theory:
  - Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.

Relation to Ehrhart Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Ehrhart Theory:
  - Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\operatorname{codeg} P := \min\left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing \right)$

Relation to Ehrhart Theory

- Ehrhart Theory:
  - ▷ Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\operatorname{codeg} P := \min\left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing\right)$
    - ▷ $\operatorname{codeg} P \leq n+1$ with equality only for a unimodular simplex

Relation to Ehrhart Theory

- Ehrhart Theory:
  - ▷ Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\operatorname{codeg} P := \min\left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing\right)$
    - ▷ $\operatorname{codeg} P \leq n+1$ with equality only for a unimodular simplex
- $\mu \leq \operatorname{codeg} P$:

Relation to Ehrhart Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Ehrhart Theory:
  - Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\operatorname{codeg} P := \min\left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing\right)$
    - $\operatorname{codeg} P \leq n+1$ with equality only for a unimodular simplex
- $\mu \leq \operatorname{codeg} P$: $\operatorname{int}(kP) \cap \mathbb{Z}^n \subseteq (kP)^{(1)}$

Relation to Ehrhart Theory

- Ehrhart Theory:
  - Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\operatorname{codeg} P := \min\left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing \right)$
    - $\operatorname{codeg} P \leq n+1$ with equality only for a unimodular simplex
- $\mu \leq \operatorname{codeg} P$: $\operatorname{int}(kP) \cap \mathbb{Z}^n \subseteq (kP)^{(1)} = k\left(P^{(\frac{1}{k})}\right)$

Relation to Ehrhart Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Ehrhart Theory:
  - Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\operatorname{codeg} P := \min\left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing\right)$
    - $\operatorname{codeg} P \leq n+1$ with equality only for a unimodular simplex
- $\mu \leq \operatorname{codeg} P$: $\operatorname{int}(kP) \cap \mathbb{Z}^n \subseteq (kP)^{(1)} = k\left(P^{(\frac{1}{k})}\right)$
- Cayley conjecture

  $\operatorname{codeg} P > \frac{n+2}{2} \Rightarrow P$ has lattice width one [Batyrev, Nill], [...]

Relation to Ehrhart Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Ehrhart Theory:
  - Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\operatorname{codeg} P := \min\left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing\right)$
    - $\operatorname{codeg} P \leq n + 1$ with equality only for a unimodular simplex
- $\mu \leq \operatorname{codeg} P$: $\operatorname{int}(kP) \cap \mathbb{Z}^n \subseteq (kP)^{(1)} = k\left(P^{(\frac{1}{k})}\right)$
- Cayley conjecture

  $\operatorname{codeg} P > \frac{n+2}{2} \Rightarrow P$ has lattice width one [Batyrev, Nill], [...]

  - X_P smooth and $\mu = \tau$ [Dickenstein, DiRocco, Piene]

Relation to Ehrhart Theory

- Ehrhart Theory:
  - Count lattice points in dilates of a polytope P: $L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\operatorname{codeg} P := \min\left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing\right)$
    - $\operatorname{codeg} P \leq n+1$ with equality only for a unimodular simplex
- $\mu \leq \operatorname{codeg} P$: $\operatorname{int}(kP) \cap \mathbb{Z}^n \subseteq (kP)^{(1)} = k\left(P^{(\frac{1}{k})}\right)$
- Cayley conjecture

  $\operatorname{codeg} P > \frac{n+2}{2} \Rightarrow P$ has lattice width one [Batyrev, Nill], [...]

  - X_P smooth and $\mu = \tau$ [Dickenstein, DiRocco, Piene]
  - X_P smooth [Dickenstein, Nill]

Relation to Ehrhart Theory

TECHNISCHE UNIVERSITÄT DARMSTADT

- Ehrhart Theory:
  - Count lattice points in dilates of a polytope P: $\quad L(k) := |kP \cap \mathbb{Z}^n|$.
  - codegree $\quad \operatorname{codeg} P := \min \left(k \mid \operatorname{int}(k \cdot P) \cap \mathbb{Z}^d \neq \varnothing \right)$
    - $\operatorname{codeg} P \leq n+1$ with equality only for a unimodular simplex
- $\mu \leq \operatorname{codeg} P$: $\quad \operatorname{int}(kP) \cap \mathbb{Z}^n \subseteq (kP)^{(1)} = k\left(P^{(\frac{1}{k})}\right)$
- Cayley conjecture

  $\operatorname{codeg} P > \frac{n+2}{2} \quad \Rightarrow \quad P$ has lattice width one $\qquad$ [Batyrev, Nill], [. . .]

  - X_P smooth and $\mu = \tau$ $\qquad$ [Dickenstein, DiRocco, Piene]
  - X_P smooth $\qquad$ [Dickenstein, Nill]
  - here: X_P Gorenstein and $\mu = \tau$

Relation to Dual Defective Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- (X_P, L_P) polarized toric variety with polytope P
- $X_P \hookrightarrow \mathbb{P}^{|A|-1}$ for $A := P \cap \mathbb{Z}^n$ projective embedding

Relation to Dual Defective Varieties

- ▷ (X_P, L_P) polarized toric variety with polytope P
- ▷ $X_P \hookrightarrow \mathbb{P}^{|A|-1}$ for $A := P \cap \mathbb{Z}^n$ projective embedding

- ▶ dual variety $X_P^\vee := \overline{\left\{H \in (\mathbb{P}^{m-1})^* \mid H \text{ tangent to } X_P \text{ at some } x \in X_P\right\}}$

Relation to Dual Defective Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- (X_P, L_P) polarized toric variety with polytope P
- $X_P \longhookrightarrow \mathbb{P}^{|A|-1}$ for $A := P \cap \mathbb{Z}^n$ projective embedding

- dual variety $X_P^\vee := \overline{\{H \in (\mathbb{P}^{m-1})^* \mid H \text{ tangent to } X_P \text{ at some } x \in X_P\}}$

- generically $X_P^\vee$ is a hypersurface
- X_P dual defective if $\dim X_P^\vee$ if $\dim X_P^\vee < |A|$

Algorithmic Discrete Mathematics polymake

Relation to Dual Defective Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- ▷ (X_P, L_P) polarized toric variety with polytope P
- ▷ $X_P \longleftrightarrow \mathbb{P}^{|A|-1}$ for $A := P \cap \mathbb{Z}^n$ projective embedding

- ▶ dual variety $X_P^\vee := \overline{\{H \in (\mathbb{P}^{m-1})^* \mid H \text{ tangent to } X_P \text{ at some } x \in X_P\}}$

- ▷ generically $X_P^\vee$ is a hypersurface
- ▶ X_P dual defective if $\dim X_P^\vee$ if $\dim X_P^\vee < |A|$

- ▶ X_P dual defective $\Longrightarrow$ $\mu(P) = \tau(P) > \frac{n+2}{2}$ [Beltrametti, Fania, Sommese]

Relation to Dual Defective Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- ▷ (X_P, L_P) polarized toric variety with polytope P
- ▷ $X_P \longrightarrow \mathbb{P}^{|A|-1}$ for $A := P \cap \mathbb{Z}^n$ projective embedding

- ▶ dual variety $X_P^\vee := \overline{\{H \in (\mathbb{P}^{m-1})^* \mid H \text{ tangent to } X_P \text{ at some } x \in X_P\}}$

- ▷ generically $X_P^\vee$ is a hypersurface
- ▶ X_P dual defective if $\dim X_P^\vee$ if $\dim X_P^\vee < |A|$

- ▶ X_P dual defective $\Longrightarrow$ $\mu(P) = \tau(P) > \frac{n+2}{2}$ [Beltrametti, Fania, Sommese]

- ▶ smooth case:

  dual defective $\iff$ $\mu(P) > \frac{n+2}{2}$ [DiRocco, Dickenstein, Piene]

  $\iff$ $\operatorname{codeg} P > \frac{n+2}{2}$ [Dickenstein, Nill]

Relation to Dual Defective Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- Theorem: If $\mu \geq \frac{3n+4}{4}$ then X_A is dual defective

Relation to Dual Defective Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- Theorem: If $\mu \geq \frac{3n+4}{4}$ then X_A is dual defective
  - ▷ P Cayley polytope of order m in dimension $r > m$

    $\Longrightarrow X_P$ has dual defect $r - m$ [Dickenstein, Feichtner, Sturmfels]

Relation to Dual Defective Varieties

TECHNISCHE UNIVERSITÄT DARMSTADT

- Theorem: If $\mu \geq \frac{3n+4}{4}$ then X_A is dual defective
  - P Cayley polytope of order m in dimension $r > m$
    $\Longrightarrow X_P$ has dual defect $r - m$ [Dickenstein, Feichtner, Sturmfels]
- Question Does $\mu > \frac{n+2}{2}$ suffice?

Relation to Dual Defective Varieties

- Theorem: If $\mu \geq \frac{3n+4}{4}$ then X_A is dual defective
  - ▷ P Cayley polytope of order m in dimension $r > m$
    $\Longrightarrow X_P$ has dual defect $r - m$ [Dickenstein, Feichtner, Sturmfels]

- Question Does $\mu > \frac{n+2}{2}$ suffice?

▷ P Cayley polytope $\Longrightarrow$ vertices of P are on two parallel hyperplanes

- X_A dual defective $\Longrightarrow$ vertices of P are on two parallel hyperplanes

[Curran, Cattani], [Esterov]

  - ▷ converse is not true, not even in smooth case