Fano Polytopes with Many Vertices

Andreas Paffenholz

joint with
Benjamin Assarf and Michael Joswig
lattice polytope $P : \iff P = \text{conv}(v_1, \ldots, v_n) \subset \mathbb{R}^d$
for lattice points $v_1, \ldots, v_n \in \mathbb{Z}^d$. 

Lattice Polytopes
Lattice Polytopes

- A lattice polytope $P$ is defined as:
  \[ P = \text{conv}(v_1, \ldots, v_n) \subset \mathbb{R}^d \]
  for lattice points $v_1, \ldots, v_n \in \mathbb{Z}^d$.

- $P$ is a Fano polytope if:
  - $0$ is in the interior of $P$
  - $v_i$ is primitive, $1 \leq i \leq n$
Toric Fano Varieties

- face fan $\Sigma$ of $P$: fan with cones $\{ \lambda v \mid \lambda \geq 0, v \in F \}$ for faces $F$ of $P$
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- $X$ Fano variety: $\iff$ $X$ normal projective variety, anticanonical divisor $K_X$ $\mathbb{Q}$-Cartier and ample
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- **$\Sigma$ complete polyhedral fan** ($\iff$ cones cover $\mathbb{R}^d$)
  - $\implies X = X_\Sigma$ associated toric variety
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  - $\rightarrow X$ projective: $\iff$
    $\Sigma$ is face fan of some polytope
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- $v_1, \ldots, v_n$ primitive generators of rays of $\Sigma$

  $\implies P := \text{conv}(v_1, \ldots, v_n)$ lattice polytope
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Toric Dictionary

- $X$ toric Fano variety with associated Fano polytope $P$

$X$ \(\mathbb{Q}\)-factorial
Weil divisors are \(\mathbb{Q}\)-Cartier

$P$ simplicial
all faces are simplices
\\( \mathcal{X} \) toric Fano variety with associated Fano polytope \( \mathcal{P} \)

- \( \mathcal{X} \) \( \mathbb{Q} \)-factorial
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- \( \mathcal{X} \) has at most terminal singularities

\( \mathcal{P} \) simplicial
- all faces are simplices

\( \mathcal{P} \) terminal
- \( \mathcal{P} \cap \mathbb{Z}^d = \text{Vert}(\mathcal{P}) \cup \{0\} \)
X toric Fano variety with associated Fano polytope $P$

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- Weil divisors are Q-Cartier

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- $P \cap \mathbb{Z}^d = \text{Vert}(P) \cup \{0\}$

$X$ Gorenstein

- $K_X$ divisor is Cartier

$P$ reflexive

- polar is lattice polytope

- polar polytope $P := \{ x | \langle x, v \rangle \leq 1, v \in \text{Vert}(P) \}$
Toric Dictionary

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  - $X$ Gorenstein
    - $K_X$ divisor is Cartier
    - $P$ reflexive
      - polar is lattice polytope
  - $X$ non-singular variety
    - $P$ smooth
      - vertices of each facet are lattice basis
Theorem (Hensley; Lagarias & Ziegler)

d, m \geq 1. Then there are, up to lattice equivalence, only finitely many

d-dimensional lattice polytopes with m interior lattice points.
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computational classifications:

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- smooth reflexive: 1, 5, 18, 124, 866, 7622, 72256, 749892, 8229721

Batyrev, Kreuzer/Nill, Øbro, Lorenz, P
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structural classifications?
simplicial, terminal and reflexive

Polytopes in Low Dimensions

$\blacktriangleright \ d = 1:\quad \quad \quad $
simplicial, terminal and reflexive

Polytopes in Low Dimensions

\[ d = 1: \]

\[ d = 2: \]

\[ P_6 \quad P_5 \quad P_{4a} \quad P_{4b} \quad P_3 \]
simplicial, terminal and reflexive Polytopes in Low Dimensions

$\boldsymbol{d = 1:}$

$\boldsymbol{d = 2:}$

$\begin{align*}
\text{P}_6 & \quad \text{P}_5 \\
\text{P}_{4a} & \quad \text{P}_{4b} \\
\text{P}_3 & \quad \text{P}_3
\end{align*}$

$\boldsymbol{d = 3:}$

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Polytopes in Low Dimensions

$d = 1$: 

$d = 2$: $P_6, P_5, P_{4a}, P_{4b}, P_3$

$d = 3$: 

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\( P, P' \) polytopes containing \( 0 \) in their interior.

\( \text{direct sum } P \oplus P' := \text{conv} \left( P \times \{0\} \cup \{0\} \times P' \right) \)

\( \text{bipyramid } \text{bipyr}(P) := \text{conv}(\{0\} \times P \cup \{e_1, -e_1\}) \)

\( \text{skew bipyramid } \text{skewbipyr}(P) := \text{conv}(\{0\} \times P \cup \{e_1, v - e_1\}) \) for a vertex \( v \) of \( P \).
General Constructions

- $P, P'$ polytopes containing $0$ in their interior.
- direct sum $P \oplus P' := \text{conv}(P \times \{0\} \cup \{0\} \times P')$
- bipyramid $\text{bipyr}(P) := \text{conv}(\{0\} \times P \cup \{e_1, -e_1\})$
- skew bipyramid
  \[
  \text{skewbipyr}(P) := \text{conv}(\{0\} \times P \cup \{e_1, v - e_1\})
  \]
  for a vertex $v$ of $P$.

- Proposition constructions preserve simplicial/terminal/reflexive
Basic Examples

(1) regular cross polytope: \[ C(d) := \text{conv}( \pm e_i \mid 1 \leq i \leq d ) \subset \mathbb{R}^d \]

(2) pseudo-Del Pezzo polytope: \[ D'(d) := \text{conv}( C(d) \cup \{1\} ) \subset \mathbb{R}^d \]

(3) Del Pezzo polytope: \[ D(d) := \text{conv}( C(d) \cup \{\pm 1\} ) \subset \mathbb{R}^d \]
Basic Examples

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Theorem [Voskrenkij & Klyachko, Ewald, Nill]

\[ P \text{ simplicial, terminal, and reflexive with antipodal pair of facets} \]
\[ \implies P \text{ is direct sum of a centrally symmetric cross polytope, (2), and (3)} \]
η-vectors

- $F$ a facet of $P$ with normal $u_F$
  
  → $\eta$-vector $\eta^F = (\eta_1, \eta_0, \eta_{-1}, \ldots)$,

  $$\eta_i := |\{ x \in \text{Vert}(P) \mid \langle u, x \rangle = i \}|$$
  
  grading on Vert$(P)$
\( \eta \)-vectors

- **F** a facet of \( P \) with normal \( \mathbf{u}_F \)
  \[ \eta^F = (\eta_1, \eta_0, \eta_{-1}, \ldots) , \]
  \[ \eta_i := |\{ \mathbf{x} \in \text{Vert}(P) \mid \langle \mathbf{u}, \mathbf{x} \rangle = i \}| \]
  grading on \( \text{Vert}(P) \)

- **F** is a special facet
  \[ \iff \mathbf{v}_P := \sum_{\mathbf{v} \in \text{Vert}(P)} \mathbf{v} \in \text{cone}(F) \]

- Proposition \( \eta_0 \leq d \) [Nil]
\( \eta \)-vectors

- \( F \) a facet of \( P \) with normal \( u_F \)
  \[ \eta^F = (\eta_1, \eta_0, \eta_{-1}, \ldots) \]
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  grading on \( \text{Vert}(P) \)

- \( F \) is a special facet
  \[ v_P := \sum_{v \in \text{Vert}(P)} v \in \text{cone}(F) \]

- Proposition \( \eta_0 \leq d \)  \[ \text{[Nill]} \]

- Theorem \( f_0 := |\text{Vert}(P)| \leq 3d \)  \[ \text{[Casagrande; Øbro]} \]
**η-vectors**

- **F** a facet of **P** with normal **u**<sub>F</sub>
  
  → **η**-vector \( \eta^F = (\eta_1, \eta_0, \eta_{-1}, \ldots) \),
  
  \[ \eta_i := |\{ x \in \text{Vert}(P) \mid \langle u, x \rangle = i \}| \] grading on \( \text{Vert}(P) \)

- **F** is a special facet
  
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- Proposition \( \eta_0 \leq d \) [Nilf]

- Theorem \( f_0 := |\text{Vert}(P)| \leq 3d \) [Casagrande; Øbro]
  
  proof: \[ 0 \leq \langle \mathbf{u}_F, \mathbf{v}_P \rangle = \eta_1 + 0 \cdot \eta_0 + (-1) \cdot \eta_{-1} + (-2) \cdot \eta_{-2} + \cdots \]
  
  \[ = d + 0 - \cdots \]
Many Vertices

\[ f_0 = 3d: \quad (a) \quad P_6^{d/2} \]
Many Vertices

- $f_0 = 3d$: \( P_{d/2} \)
- Theorem: There are no other cases. [Casagrande]
Many Vertices

▶ $f_0 = 3d$:

(a) $P_6^{\oplus d/2}$

▶ Theorem There are no other cases. [Casagrande]

▶ $f_0 = 3d - 1$:

(b) $P_5 \oplus P_6^{\oplus d/2 - 1}$

(c) proper or skew bipyramid over $P_6^{\oplus (d-1)/2}$
Many Vertices

- $f_0 = 3d$:  
  
  - (a) $P_6^{\oplus d/2}$

- Theorem There are no other cases.  
  [Casagrande]

- $f_0 = 3d - 1$:  
  
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Many Vertices

- $f_0 = 3d$:
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  - (c) proper or skew bipyramid over $P_6^\oplus (d-1)/2$
- Theorem There are no other cases. [Øbro & Nill]

- $f_0 = 3d - 2$:
  - (d) $P_5^2 \oplus P_6^\oplus d/2 - 2$
  - (e) $D(4) \oplus P_6^\oplus d/2 - 2$
  - (f) proper or skew bipyramid over (b) or (c)
  - (g) double proper or skew bipyramid over (a)
Many Vertices

- $f_0 = 3d$: (a) $P_6^{d/2}$

- Theorem: There are no other cases. [Casagrande]

- $f_0 = 3d - 1$: (b) $P_5 \oplus P_6^{d/2 - 1}$
  (c) proper or skew bipyramid over $P_6^{(d-1)/2}$

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- $f_0 = 3d - 2$: (d) $P_5^2 \oplus P_6^{d/2 - 2}$
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  (f) proper or skew bipyramid over (b) or (c)
  (g) double proper or skew bipyramid over (a)

- Theorem: There are no other cases. [Assarf, Joswig, and P]
Many Vertices: Algebraic version

- $X$ $d$-dimensional terminal $\mathbb{Q}$-factorial Gorenstein toric Fano variety
- in this case: Picard number $\rho(X) = f_0 - d$ for $f_0 =$ number of vertices
- $S$ toric del Pezzo surface with $\rho_S = 4$, $\mathbb{P}^2$ blown up in three points

**Corollary**

$$\rho(X) \geq 2d - 2.$$  
Then $X$ is a product of $S^{d/2 - k}$ with a toric Fano $k$-fold for $k \leq 4$. 
Many Vertices: proof sketch

- $P$ simplicial, terminal, and reflexive $d$-polytope
- classify $\eta$-vectors for a special facet $F$
- $v_P := \sum_{v \in \text{Vert}(P)} v$
- $\ell$: height of $v_P$ above $F$
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(a) (b) (c) (d) (e) (f) (g) all facets special
Many Vertices: proof sketch

- $P$ simplicial, terminal, and reflexive $d$-polytope
- classify $\eta$-vectors for a special facet $F$
- $v_P := \sum_{v \in \text{Vert}(P)} v$
- $\ell$: height of $v_P$ above $F$
- consider cases separately

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  - e.g., all $\eta^F$ of type (g) $\implies$ polytope is centrally symmetric

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- consider cases separately
  - e.g., all $\eta^F$ of type (g) $\implies$ polytope is centrally symmetric
  - (d) does not occur $\iff$ look at adjacent facet
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- $v_P := \sum_{v \in \text{Vert}(P)} v$
- $\ell$: height of $v_P$ above $F$

<table>
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<td>d</td>
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<tr>
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<td>d</td>
<td>d-1</td>
<td>d</td>
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<td>d-1</td>
<td>d-2</td>
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<tr>
<td>$\eta_{-1}$</td>
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<td>d-3</td>
<td>d-1</td>
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<td>d-4</td>
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- consider cases separately
  - e.g., all $\eta^F$ of type (g) $\implies$ polytope is centrally symmetric
  - (d) does not occur $\implies$ look at adjacent facet
- show that polytopes are either
  - direct sum of $P_6$ with $(d-2)$-polytope
  - (skew) bipyramid over $(d-1)$-polytope
Many Vertices: proof sketch

- $P$ simplicial, terminal, and reflexive $d$-polytope
- classify $\eta$-vectors for a special facet $F$
- $v_P := \sum_{v \in \text{Vert}(P)} v$
- $\ell$: height of $v_P$ above $F$

<table>
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(a) (b) (c) (d) (e) (f) (g)

all facets special

- consider cases separately
  - e.g., all $\eta^F$ of type (g) $\implies$ polytope is centrally symmetric
  - (d) does not occur $\iff$ look at adjacent facet
  - show that polytopes are either
    - direct sum of $P_6$ with $(d - 2)$-polytope
    - (skew) bipyramid over $(d - 1)$-polytope
  - exact types via induction
Can we continue?

\[ f_0 = 3d - 3 \? \]

- \( R := \text{skew bipyramid over } P_6 \rightarrow 8 \text{ vertices and 12 facets} \)
- \( P := R^\oplus 3 \rightarrow 3 \cdot 8 = 3 \cdot 9 - 3 \text{ vertices in dimension } d = 9 \)
- \( P \) is not a (skew) bipyramid over a sum of \( P_5 \) and \( P_6 \)
Can we continue?

- $f_0 = 3d - 3$?

  - $R :=$ skew bipyramid over $P_6$  
    $\rightarrow$ 8 vertices and 12 facets
  - $P := R^{\oplus 3}$  
    $\rightarrow$ 3 · 8 = 3 · 9 − 3 vertices in dimension $d = 9$

  - $P$ is not a (skew) bipyramid over a sum of $P_5$ and $P_6$

- **Theorem** [Assarf, Joswig, P]

  $P$ terminal, simplicial, reflexive $d$-polytope with $3d - 2$ vertices

  Then $P$ is $\triangleright P_5^2 \oplus P_6^{\oplus d/2-2}$, or

  $\triangleright D(4) \oplus P_6^{\oplus d/2-2}$, or

  $\triangleright$ (double) proper/skew bipyramid over $P_6^{\oplus k}$ for suitable $k$
Can we continue?

$\triangledown f_0 = 3d - 3$?

- $R :=$ skew bipyramid over $P_6 \rightarrow 8$ vertices and 12 facets
- $P := R^{\oplus 3} \rightarrow 3 \cdot 8 = 3 \cdot 9 - 3$ vertices in dimension $d = 9$
- $P$ is not a (skew) bipyramid over a sum of $P_5$ and $P_6$

- Theorem
  
  $P$ terminal, simplicial, reflexive $d$-polytope with $3d - 2$ vertices
  
  Then $P = Q \oplus P_6^{\oplus k}$ for suitable $k$ and dim $Q \leq 4$. 

[Assarf, Joswig, P]
Can we continue?

\( f_0 = 3d - 3 \) ?

- \( R := \text{skew bipyramid over } P_6 \)
  \( \rightarrow \) 8 vertices and 12 facets

- \( P := R \oplus^3 \rightarrow 3 \cdot 8 = 3 \cdot 9 - 3 \) vertices in dimension \( d = 9 \)

- \( P \) is not a (skew) bipyramid over a sum of \( P_5 \) and \( P_6 \)

Theorem [Assarf, Joswig, P]

\( P \) terminal, simplicial, reflexive \( d \)-polytope with \( 3d - 2 \) vertices

Then \( P = Q \oplus P_6 \oplus^k \) for suitable \( k \) and \( \dim Q \leq 4 \).

Conjecture [Assarf, Joswig, P]

\( P \) smooth Fano \( d \)-polytope with \( 3d - k \) vertices, \( k \leq d/3 \)

Then \( P = Q \oplus P_6^l \) for \( \dim Q \leq 3k \) and appropriate \( l \).