Fano Polytopes with Many Vertices

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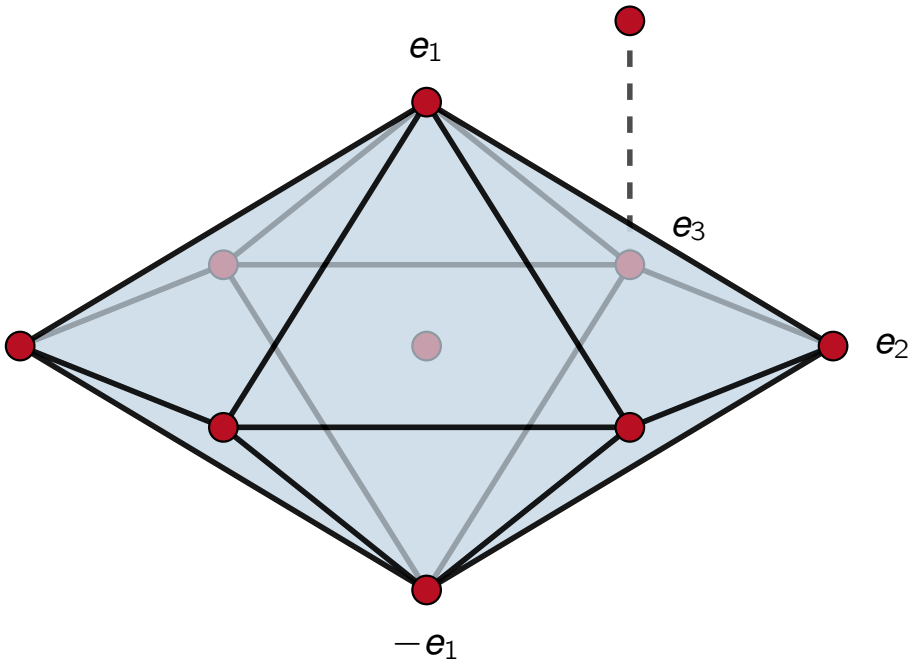
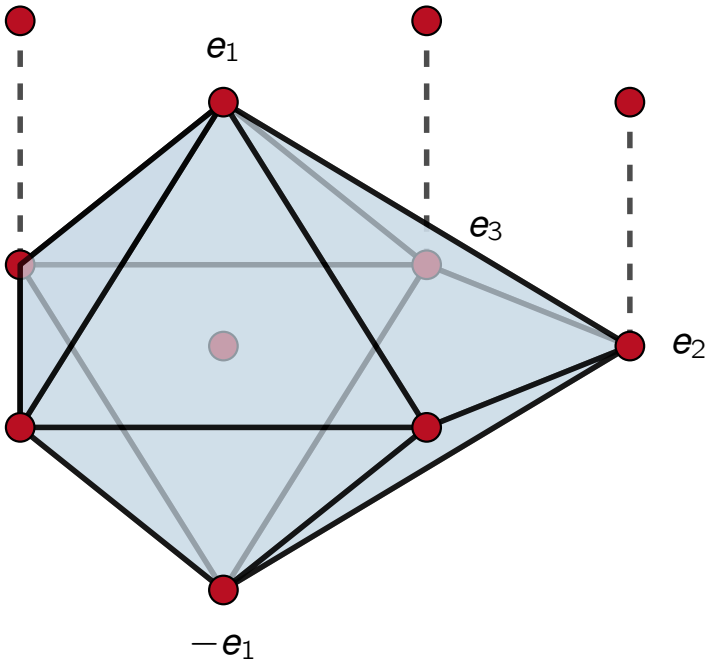
Andreas Paffenholz

joint with

Benjamin Assarf and Michael Joswig





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Lattice Polytopes

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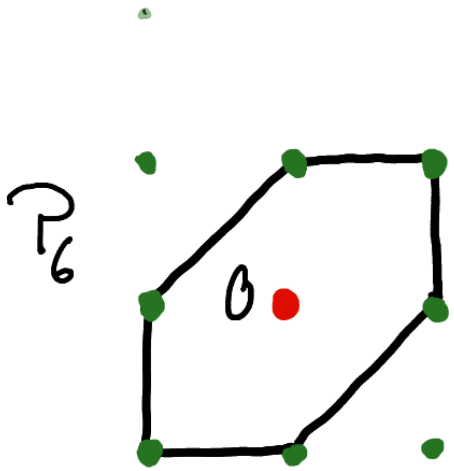
- lattice polytope $P :\Longleftrightarrow \quad P = \operatorname{conv}(\mathbf{v}_1, \dots, \mathbf{v}_n) \subset \mathbb{R}^d$
  for lattice points $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{Z}^d$.



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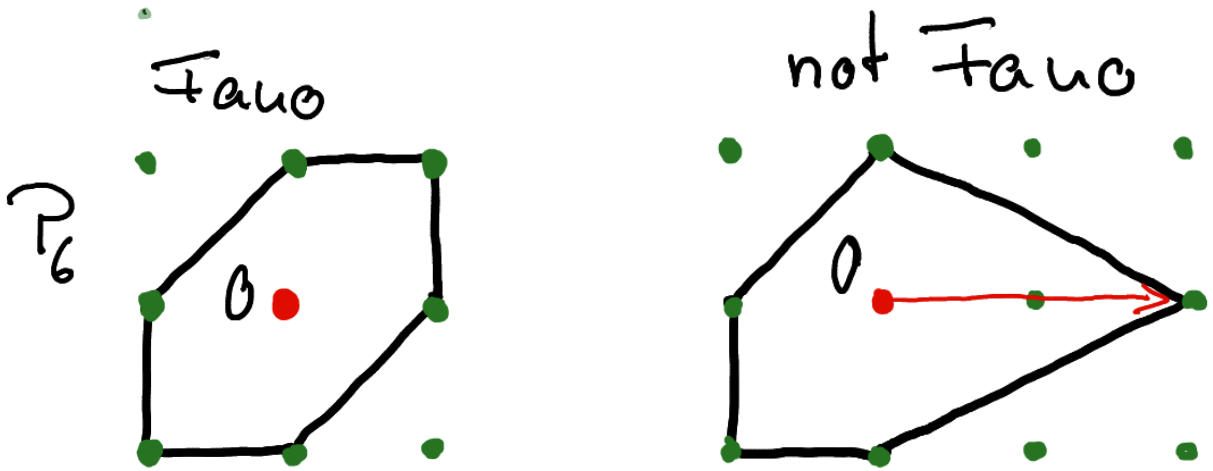
  for lattice points $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{Z}^d$.
- P is a Fano polytope $:\Longleftrightarrow$
  - $\mathbf{0}$ is in the interior of P
  - $\mathbf{v}_i$ is primitive, $1 \leq i \leq n$



Toric Fano Varieties

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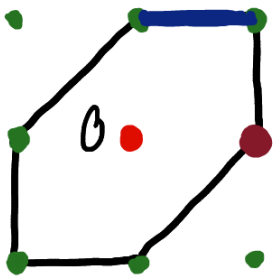

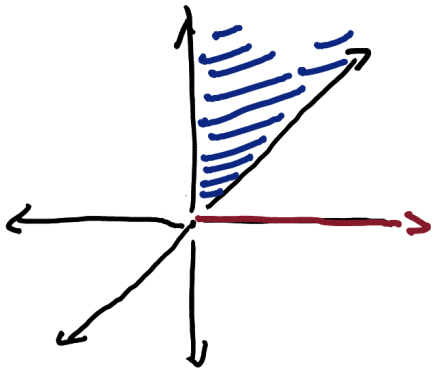

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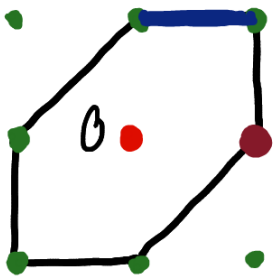

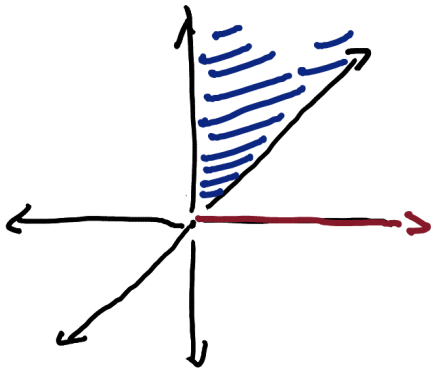

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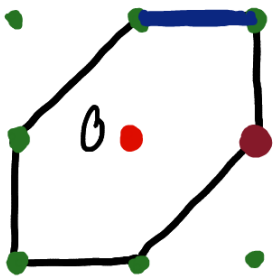
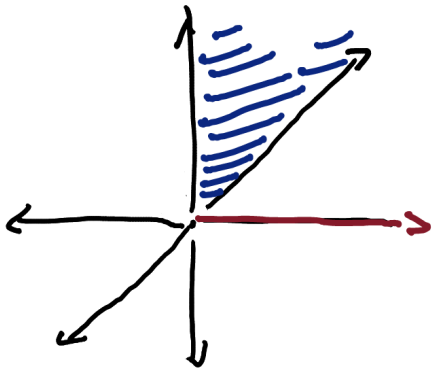
$\longrightarrow$ $X = X_\Sigma$ associated toric variety





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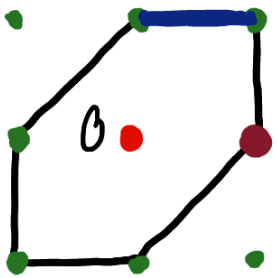

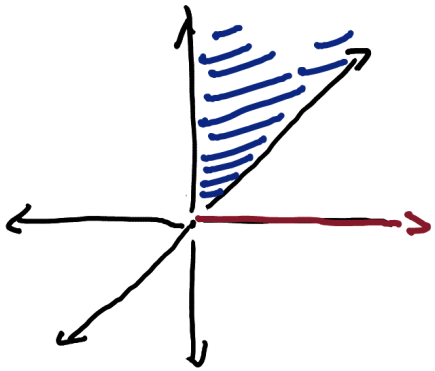

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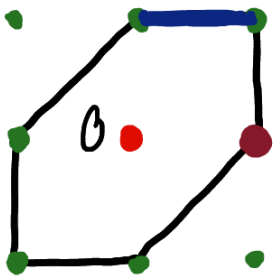
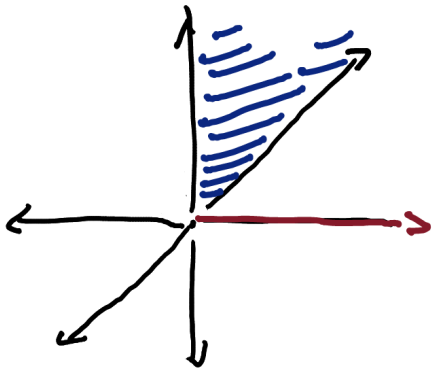


▷ $\mathbf{v}_1, \dots, \mathbf{v}_n$ primitive generators of rays of Σ

$\longrightarrow$ $P := \operatorname{conv}(\mathbf{v}_1, \dots, \mathbf{v}_n)$ lattice polytope



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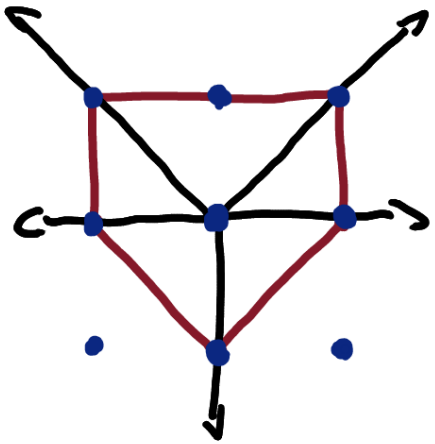

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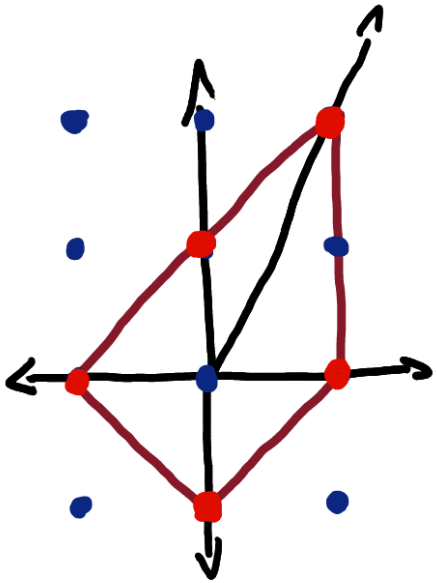

Toric Dictionary

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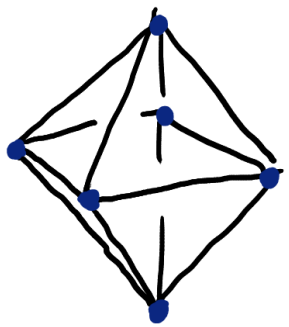
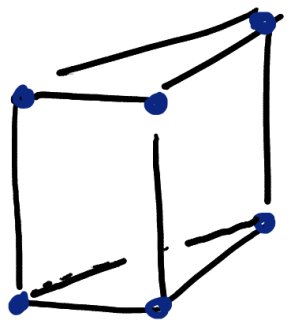
- X toric Fano variety with associated Fano polytope P

X $\mathbb{Q}$-factorial
Weil divisors
are $\mathbb{Q}$-Cartier

$\longleftrightarrow$

P simplicial
all faces are simplices





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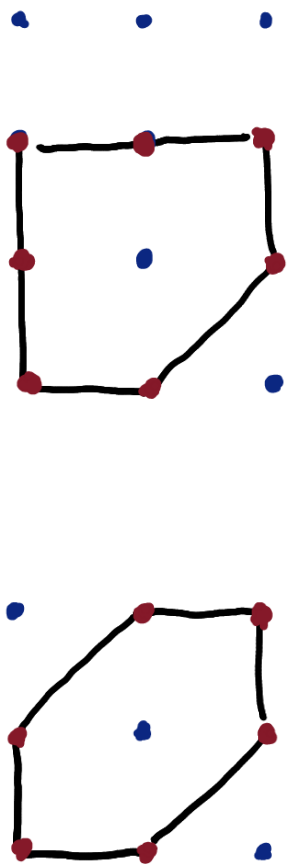

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X has at most
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P terminal
$P \cap \mathbb{Z}^d \;=\; \mathrm{Vert}(P) \cup \{\mathbf{0}\}$

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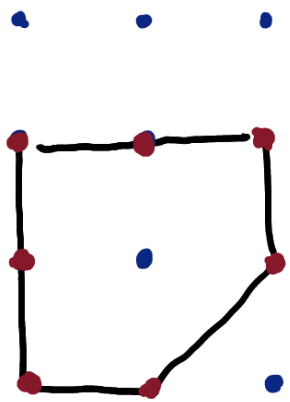

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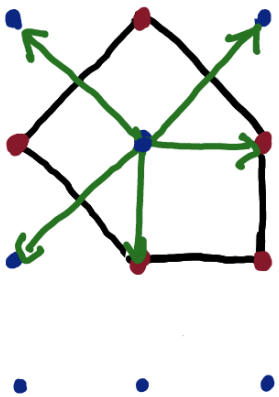
P reflexive
polar is lattice polytope



▶ polar polytope $P \;:=\; \{\, \mathbf{x} \mid \langle \mathbf{x}, \mathbf{v} \rangle \;\leq\; 1 \quad \mathbf{v} \in \mathrm{Vert}(P) \,\}$

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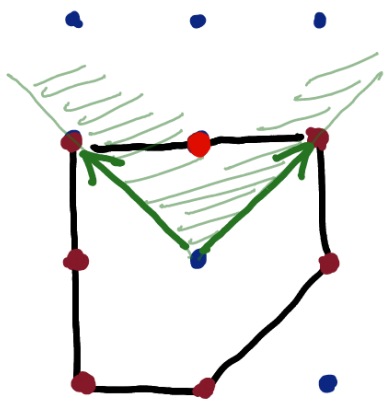

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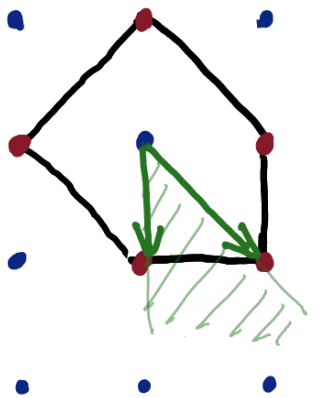

X Gorenstein
K_X divisor is Cartier

$\longleftrightarrow$

P reflexive
polar is lattice polytope

▶ X non-singular variety

$\longleftrightarrow$

P smooth
vertices of each facet
are lattice basis

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Finiteness and Classification

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- Theorem (Hensley; Lagarias & Ziegler)

  $d, m \geq 1$. Then there are, up to lattice equivalence, only finitely many d-dimensional lattice polytopes with m interior lattice points.

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- structural classifications?

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simplicial, terminal and reflexive Polytopes in Low Dimensions

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- $d = 1$: 

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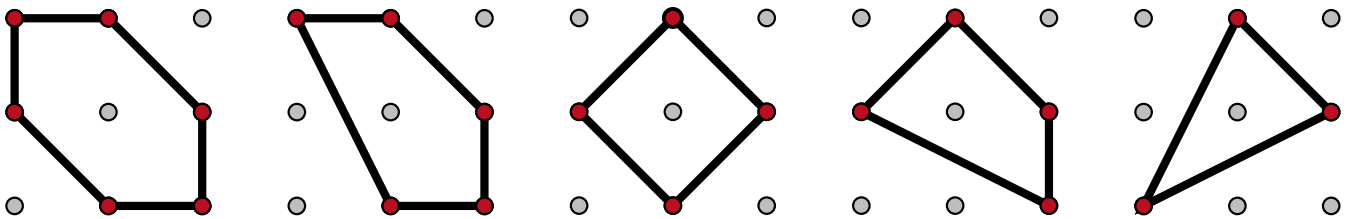
- $d = 2$: 

P_6 P_5 P_{4a} P_{4b} P_3

simplicial, terminal and reflexive Polytopes in Low Dimensions

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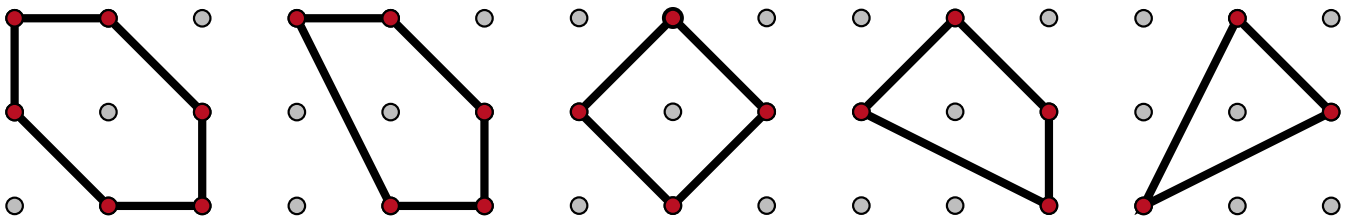

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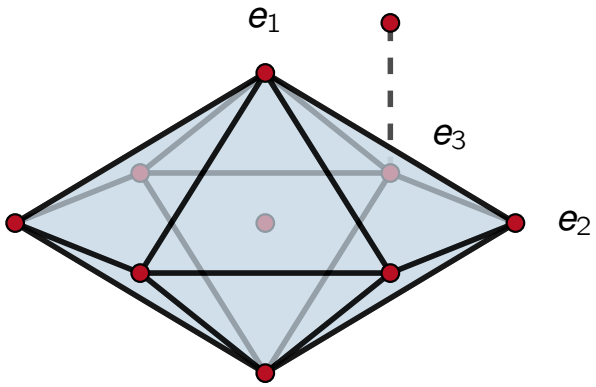
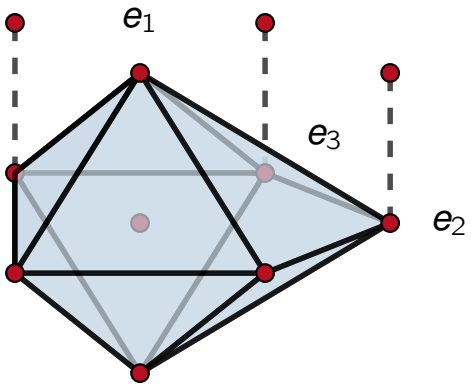
- $d = 3$:





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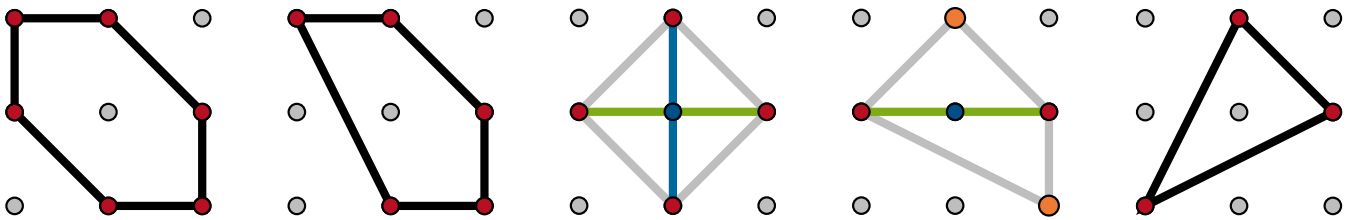
- $d = 2$: 

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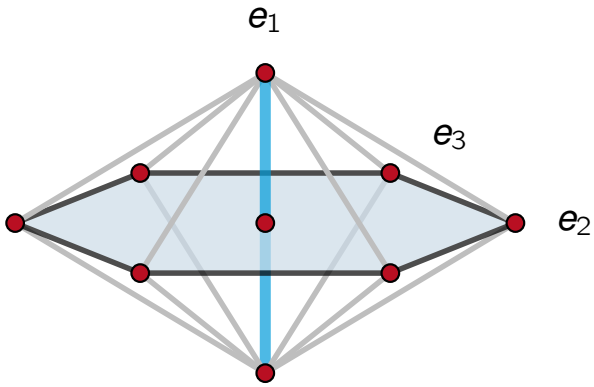
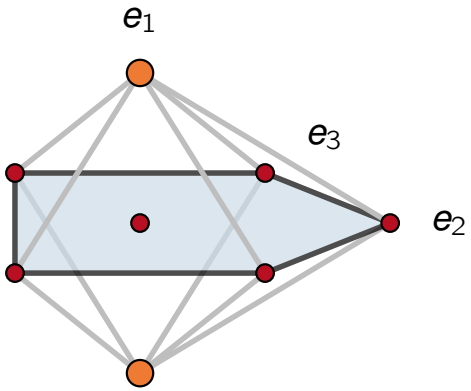
- $d = 3$: 



General Constructions

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- P, P' polytopes containing $\mathbf{0}$ in their interior.
- direct sum $P \oplus P' \;:=\; \operatorname{conv}\big(\, P \times \{\mathbf{0}\} \;\cup\; \{\mathbf{0}\} \times P' \,\big)$
- bipyramid $\operatorname{bipyr}(P) \;:=\; \operatorname{conv}(\{0\} \times P \cup \{e_1, -e_1\})$
- skew bipyramid

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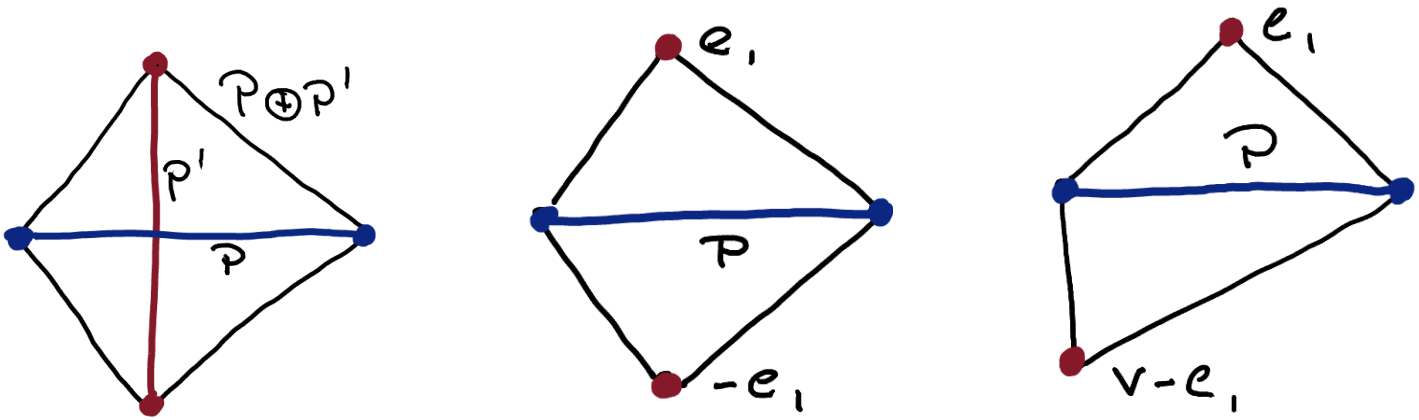

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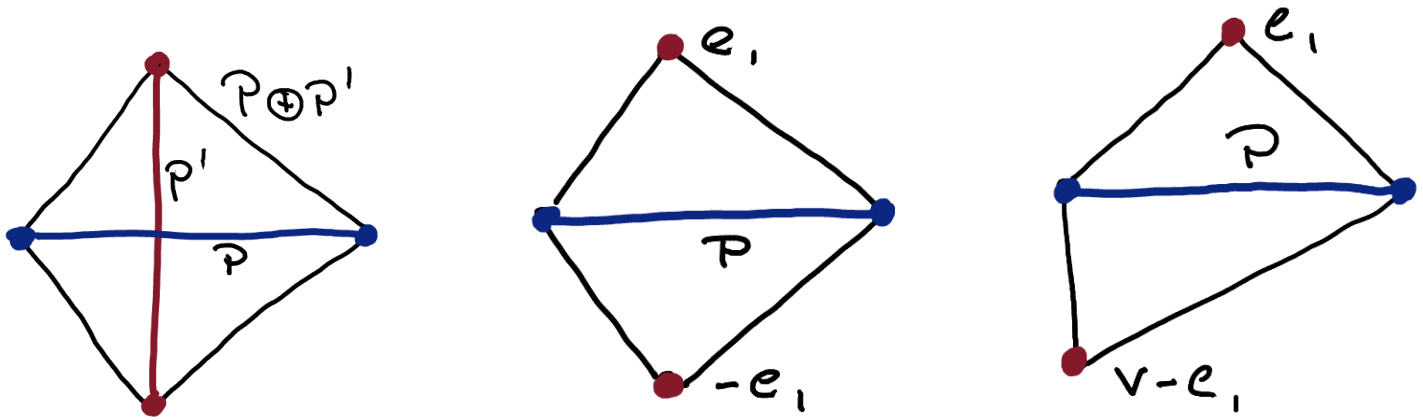


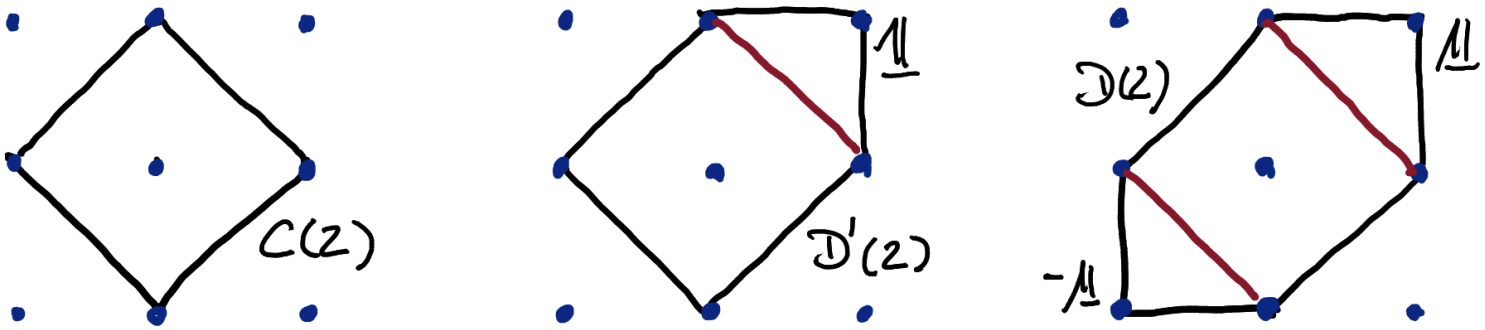
- Proposition constructions preserve simplicial/terminal/reflexive

Basic Examples

(1) regular cross polytope: $C(d) \;:=\; \mathrm{conv}(\,\pm\mathbf{e}_i \mid 1 \le i \le d\,) \;\subset\; \mathbb{R}^d$

(2) pseudo-Del Pezzo polytope: $D'(d) \;:=\; \mathrm{conv}\,(\,C(d) \,\cup\, \{\mathbf{1}\}\,) \;\subset\; \mathbb{R}^d$

(3) Del Pezzo polytope: $D(d) \;:=\; \mathrm{conv}\,(\,C(d) \,\cup\, \{\pm\mathbf{1}\}\,) \;\subset\; \mathbb{R}^d$



Basic Examples

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(1) regular cross polytope: $C(d) := \operatorname{conv}(\pm\mathbf{e}_i \mid 1 \le i \le d) \subset \mathbb{R}^d$

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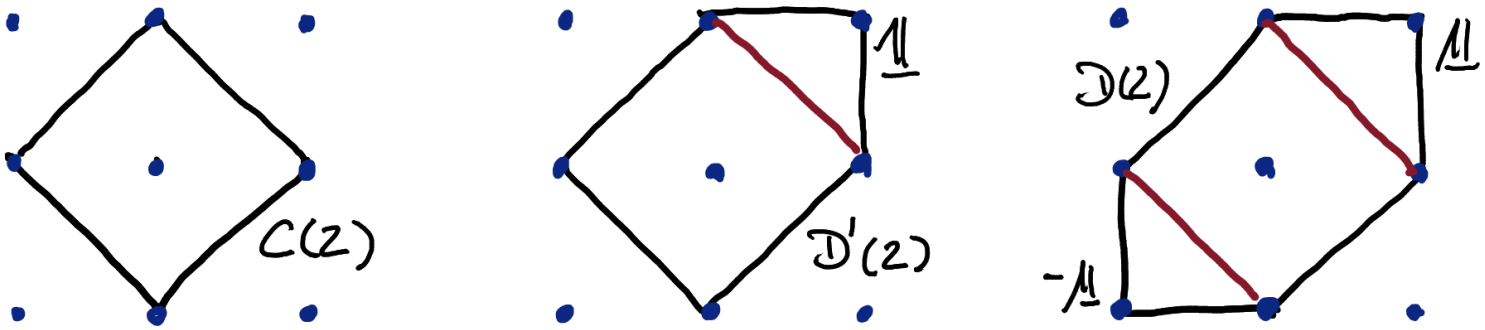

▶ Theorem [Voskrenkij&Klyachko, Ewald, Nill]

P simplicial, terminal, and reflexive with antipodal pair of facets

$\Longrightarrow$ P is direct sum of a centrally symmetric cross polytope, (2), and (3)

η-vectors

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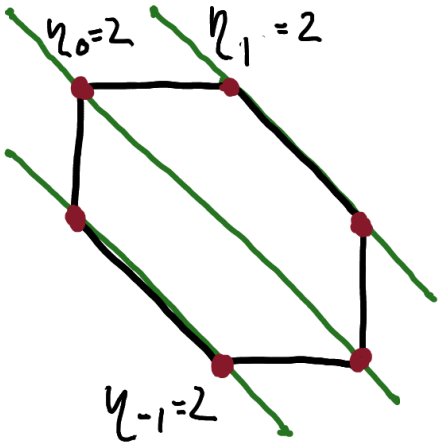
- F a facet of P with normal $\mathbf{u}_F$

$$\longrightarrow \eta\text{-vector } \eta^F \;=\; (\eta_1, \eta_0, \eta_{-1}, \dots)\,,$$

$$\eta_i \;:=\; |\{\, \mathbf{x} \in \mathrm{Vert}(P) \mid \langle \mathbf{u}, \mathbf{x} \rangle \;=\; i \,\}| \qquad \text{grading on } \mathrm{Vert}(P)$$



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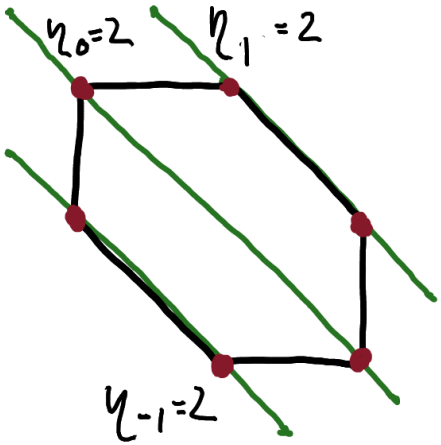

- F is a special facet

  $:\Longleftrightarrow \quad \mathbf{v}_P \;:=\; \sum_{v \in \mathrm{Vert}(P)} \mathbf{v} \;\in\; \mathrm{cone}(F)$

- Proposition $\eta_0 \;\leq\; d$ [Nill]

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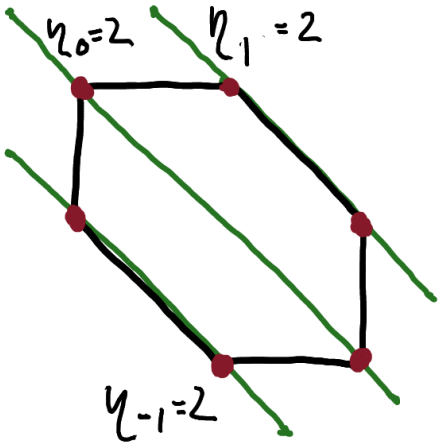

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- Theorem $f_0 \;:=\; |\mathrm{Vert}(P)| \;\leq\; 3d$ [Casagrande; Øbro]

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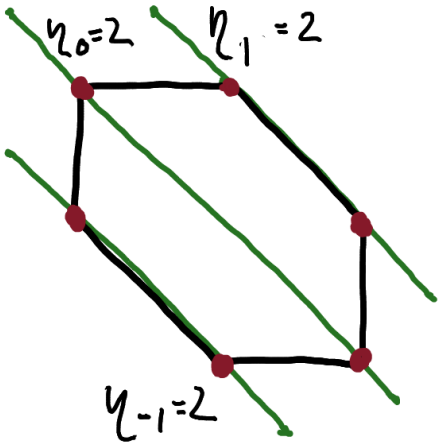

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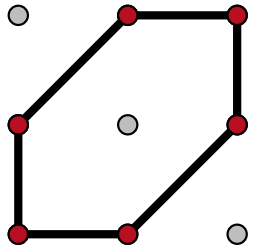
- Theorem $f_0 \;:=\; |\mathrm{Vert}(P)| \;\leq\; 3d$ [Casagrande; Øbro]

  proof: $0 \;\leq\; \langle \mathbf{u}_F, \mathbf{v}_P \rangle \;=\; \eta_1 \;+\; 0 \cdot \eta_0 \;+\; (-1) \cdot \eta_{-1} \;+\; (-2) \cdot \eta_{-2} \;+\; \cdots$

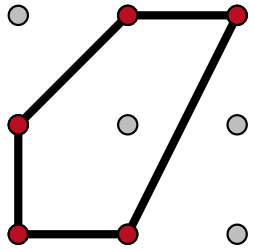
  $= \; d \;+\; 0 \qquad - \cdots$

Many Vertices

- $f_0 = 3d$: **(a)** $P_6^{\oplus d/2}$



P_6



P_5

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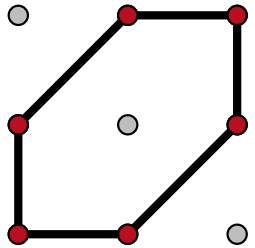

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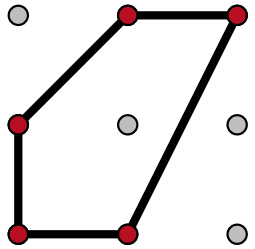

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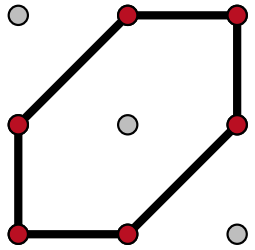
- $f_0 = 3d - 1$: **(b)** $P_5 \oplus P_6^{\oplus d/2 - 1}$

  (c) proper or skew bipyramid over $P_6^{\oplus (d-1)/2}$



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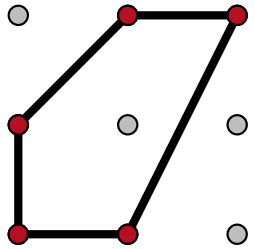

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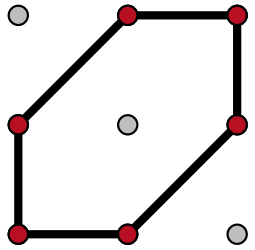

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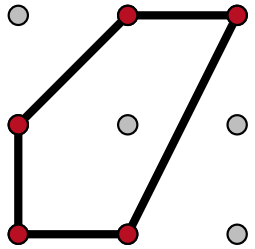

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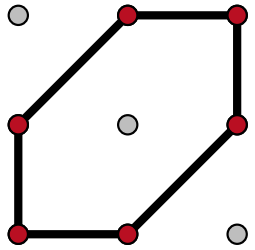

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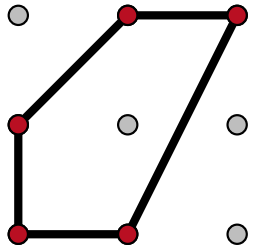

P_5

- Theorem There are no other cases. [Øbro & Nill]
- $f_0 = 3d - 2$: **(d)** $P_5^2 \oplus P_6^{\oplus d/2 - 2}$
  (e) $D(4) \oplus P_6^{\oplus d/2 - 2}$
  (f) proper or skew bipyramid over **(b)** or **(c)**
  (g) double proper or skew bipyramid over **(a)**

Many Vertices

TECHNISCHE UNIVERSITÄT DARMSTADT

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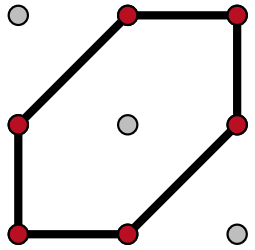

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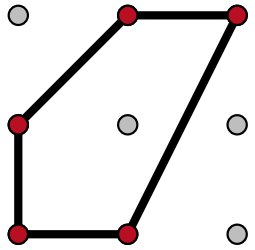

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- Theorem There are no other cases. [Assarf, Joswig, and P]

Many Vertices: Algebraic version

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- X d-dimensional terminal $\mathbb{Q}$-factorial Gorenstein toric Fano variety
- in this case: Picard number $\rho(X) \;=\; f_0 - d$ for $f_0 \;=\;$ number of vertices

- S toric del Pezzo surface with $\rho_S \;=\; 4$, $\mathbb{P}^2$ blown up in three points

- Corollary

  $\rho(X) \;\geq\; 2d - 2.$

  Then X is a product of $S^{d/2-k}$ with a toric Fano k-fold for $k \leq 4$

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Many Vertices: proof sketch

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- P simplicial, terminal, and reflexive d-polytope
- classify η-vectors for a special facet F
- $\mathbf{v}_P := \sum_{\mathrm{v} \in \mathrm{Vert}(P)} \mathbf{v}$
- ℓ: height of $\mathbf{v}_P$ above F

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ℓ	2	1	1	0	0	0	0
η_1	d	d	d	d	d	d	d
η_0	d	d	$d-1$	d	d	$d-1$	$d-2$
η_{-1}	$d-2$	$d-3$	$d-1$	$d-3$	$d-4$	$d-2$	d
η_{-2}	0	1	0	0	2	1	0
η_{-3}	0	0	0	1	0	0	0
	(a)	**(b)**	**(c)**	**(d)**	**(e)**	**(f)**	**(g)**

all facets special

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- consider cases separately

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- consider cases separately

▷ *e.g.*, ▷ all η^F of type **(g)** $\Longrightarrow$ polytope is centrally symmetric

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▷ **(d)** does not occur $\longleftarrow$ look ar adjacent facet

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all facets special

- consider cases separately
  - ▷ *e.g.*, ▷ all η^F of type **(g)** $\Longrightarrow$ polytope is centrally symmetric
    - ▷ **(d)** does not occur $\longleftarrow$ look ar adjacent facet
- show that polytopes are either
  - ▷ direct sum of P_6 with $(d-2)$-polytope
  - ▷ (skew) bipyramid over $(d-1)$-polytope

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- exact types via induction

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Can we continue?

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- $f_0 = 3d - 3$?
  - R := skew bipyramid over P_6

    $\longrightarrow$ 8 vertices and 12 facets
  - $P := R^{\oplus 3}$ $\longrightarrow$ $3 \cdot 8 = 3 \cdot 9 - 3$ vertices in dimension $d = 9$
  - P is not a (skew) bipyramid over a sum of P_5 and P_6

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- Theorem [Assarf, Joswig, P]

  P terminal, simplicial, reflexive d-polytope with $3d - 2$ vertices

  Then P is
  - $P_5^2 \oplus P_6^{\oplus d/2-2}$, or
  - $D(4) \oplus P_6^{\oplus d/2-2}$, or
  - (double) proper/skew bipyramid over $P_6^{\oplus k}$ for suitable k

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- Theorem [Assarf, Joswig, P]

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  Then $P = Q \oplus P_6^{\oplus k}$ for suitable k and $\dim Q \leq 4$.

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- Theorem [Assarf, Joswig, P]

  P terminal, simplicial, reflexive d-polytope with $3d - 2$ vertices

  Then $P = Q \oplus P_6^{\oplus k}$ for suitable k and $\dim Q \leq 4$.
- Conjecture [Assarf, Joswig, P]

  P smooth Fano d-polytope with $3d - k$ vertices, $k \leq d/3$

  Then $P = Q \oplus P_6^{l}$ for $\dim Q \leq 3k$ and appropriate l.

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