Finiteness of the polyhedral $\mathbb{Q}$-codegree spectrum

Andreas Paffenholz

AMS Western Sectional Meeting, San Francisco
Polytopes and Lattices

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= \{ x \mid \langle a_i, x \rangle \leq b_i, \ 1 \leq i \leq m \}
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bounded intersection of finitely many affine half spaces.

[Minkowski-Weyl(-Farkas) \( \sim \)1900]
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Ehrhart function: $g_P(k) := |kP \cap \mathbb{Z}^d|$
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Ehrhart Polynomials, Degree, and Codegree

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- Theorem: $g_P(k)$ is a polynomial of degree $d$

$g_P(t) = \frac{5}{2} t^2 + \frac{5}{2} t + 1$

1, 6, 16, 31, …

[Ehrhart 1962]
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- **\( h^* \)-polynomial**: numerator of rational generating function:

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  for nonnegative integers \( h^*_r, h^*_{r-1}, \ldots, h^*_0 \) [Stanley, 1980]

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► classifications for codegree \( \text{codeg} = 1, 2 \)

(up to lattice pyramids) \[Batyrev, Nill, Treutlein\]
Cayley-Polytopes

Structure of polytopes with large codegree?

polytopes with lattice projection onto unit simplex \( (:= \text{ Cayley polytopes}) \)
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- Conjecture:
  \[ \text{codeg} > \frac{d + 2}{2} \quad \Rightarrow \quad \text{P nontrivially projects onto lattice simplex} \]

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- true for smooth polytopes [Dickenstein, Nill, ’10]
  (smooth := rays of each normal cone are lattice basis)
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Theorem: \([\text{Haase, Nill, Payne}]\)

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2d > (\deg P)^2 + 19 \deg P - 4 \\
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(\deg P)^2 + 19 \deg P - 4 \gg 2 \deg P - 1
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\( P := \{ x \mid \langle a_i, x \rangle \leq b_i, \ 1 \leq i \leq m \} \) lattice polytope
with \( a_i, b_i \) integer, entries of \( a_i \) coprime (\( a_i \) is primitive)
Adjunction and $\mathbb{Q}$-Codegree

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- $\mathbb{Q}$-codegree: $\text{codeg}_\mathbb{Q} P := \left[ \sup(c \mid P^{(c)} \neq \emptyset) \right]^{-1} = \left[ \sup(c \mid \dim P^{(c)} \neq \dim P) \right]^{-1}$

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[Di Rocco, Haase, Nill, P ‘14]
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- almost best possible:
  $\text{codeg}_\mathbb{Q} 2\Delta_2 = \frac{3}{2} = \frac{2+1}{2}$

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- almost best possible:
  
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- Conjecture: $\text{codeg}_\mathbb{Q} P > \frac{d + 1}{2}$ suffices
rational fan $\rightarrow$ rays have minimal integral generators $a_i \in \mathbb{Z}^d$
Toric Geometry

$\Sigma$ rational fan $\rightarrow$ rays have minimal integral generators $a_i \in \mathbb{Z}^d$

fan $\Sigma$ $\leftrightarrow$ projective toric variety $X$

[Danilov '78, Fulton '93, Oda '88, Cox '95]
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ray \( \tau \) \[\leftrightarrow\] torus invariant divisor \( D_\tau \)
A rational fan $\Sigma$ is associated with a projective toric variety $X$. Each ray $\tau$ corresponds to a torus invariant divisor $D_\tau$. A Weil divisor $D$ is defined as $D := \sum b_\tau D_\tau$. 

[Danilov '78, Fulton '93, Oda '88, Cox '95]
Toric Geometry

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**D ample und Q-Cartier**:

$\iff$ $P_D := \{x \mid \langle a_\tau, x \rangle \leq b_\tau\}$

polytope with normal fan $\Sigma$
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polytope with normal fan $\Sigma$

- $D$ big: $X$ and $P_D$ have the same dimension

- $X$ is smooth (i.e. is a manifold):
  - minimal generators of every cone are a lattice basis
(X, L) polarized toric variety, $K_X$ canonical divisor on $X$
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family of adjoint divisors: $L + t \cdot K_X$
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(unnormalized) spectral value:
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(X, L) toric $\rightarrow$ lattice polytope $P = \{x \mid \langle a_i, x \rangle \leq b_i \}$

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**Theorem:** \(\sigma \geq (d+2)/2 \implies\) morphism \(\pi : \mathbb{P}(H_0 \oplus H_1 \oplus \cdots \oplus H_m) \rightarrow X\)

\(H_i\) line bundles over toric variety of dim \(\leq 2(n + 1 - \mu)\)

[Di Rocco, Haase, Nill, P. ’14]
\[ S_d := \left\{ \mu \mid \exists (X, L) \text{ smooth proj.} \right. \]
\[ \left. d\text{-dim. pol. variety} \right. \]
\[ \left. \text{with } \mu = \mu(L) \right. \]
The Spectrum Conjecture of Fujita

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**Spectrum Conjecture:**

\[ d \geq 1, \varepsilon > 0 ; \text{ Then} \]
\[ \{ \mu \in S_d \mid \mu > \varepsilon \} \text{ is a finite set.} \]

[Fujita '92,'96]
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▷ more generally: polytopes with \( \mathbb{Q} \)-Gorenstein normal fan

(\( \iff \) \( \mathbb{Q} \)-Gorenstein toric polarized varieties)
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**Proof** has two steps: 

- only finitely many normal fans  
- \( S_d^p \) finite for fixed normal fan
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**Proof** has two steps:

▶ only finitely many normal fans

▶ \( S_d^p \) finite for fixed normal fan

▶ first step essentially uses

**Theorem:** \( m, d \geq 1 \)

Up to lattice equivalence, there are only finitely many lattice \( d \)-polytopes with \( m \) interior lattice points.

[Hensley, Lagarias & Ziegler, Pikhurko]