# Finiteness of the polyhedral $\mathbb{Q}$-codegree spectrum

Andreas Paffenholz

AMS Western Sectional Meeting, San Francisco

# Polytopes and Lattices

TECHNISCHE UNIVERSITÄT DARMSTADT

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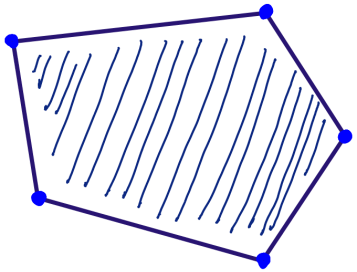



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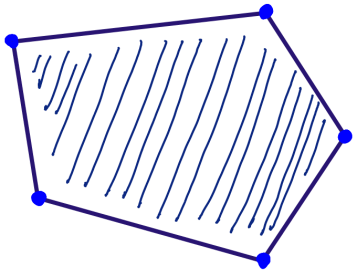

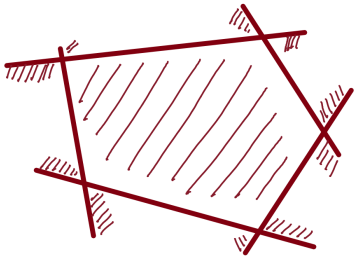

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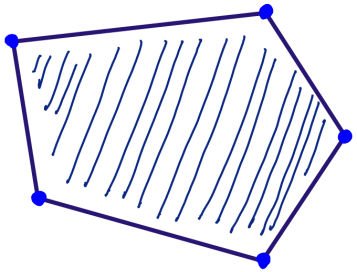

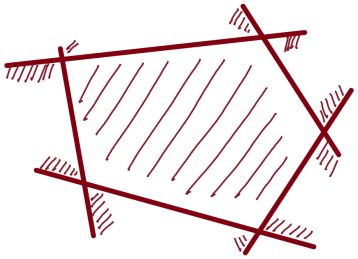

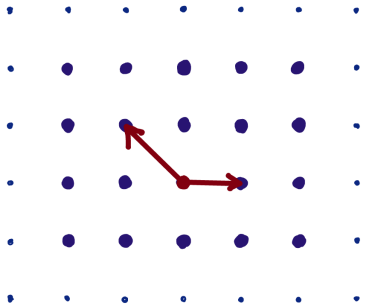

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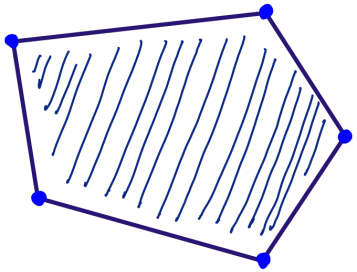

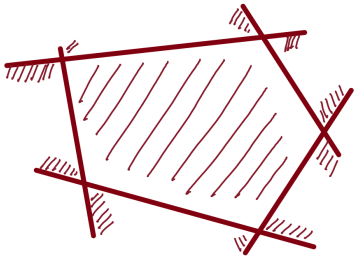

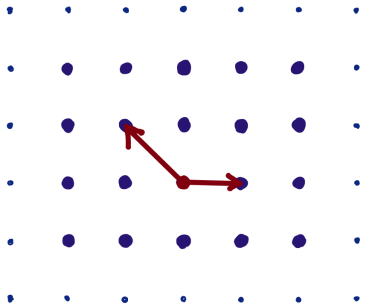

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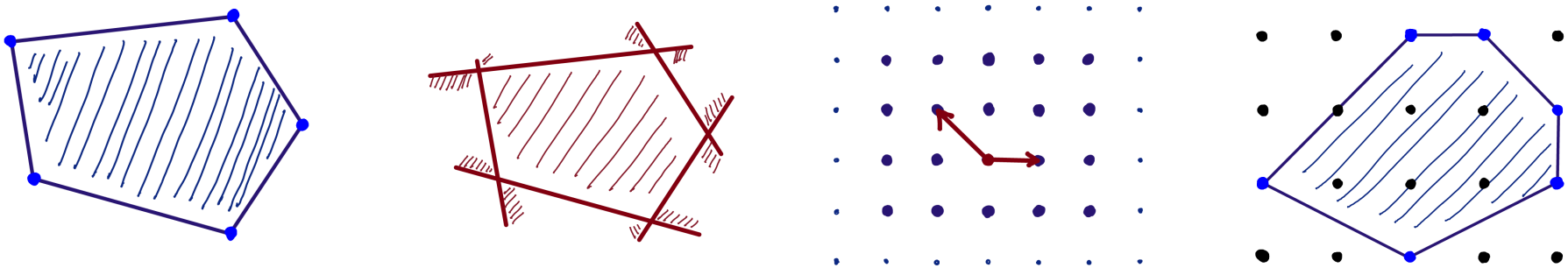



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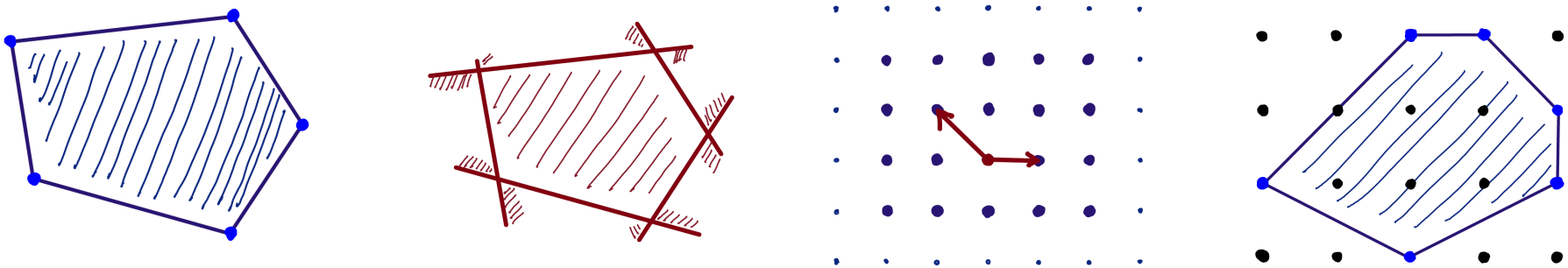

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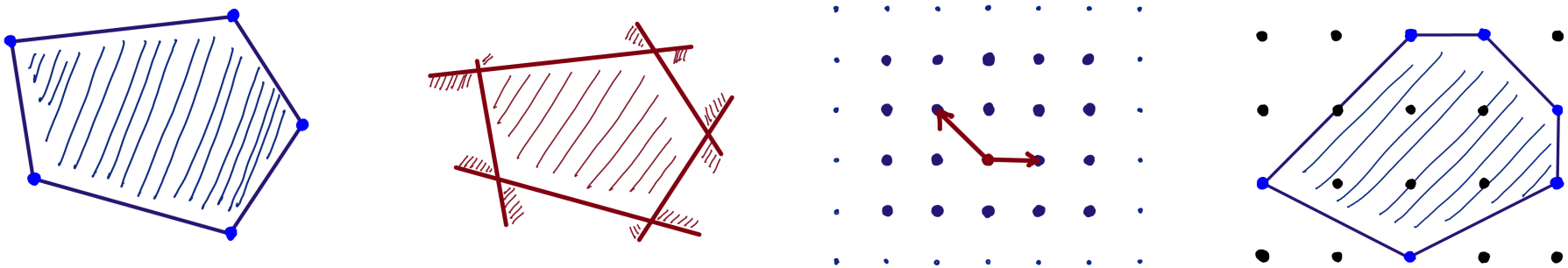

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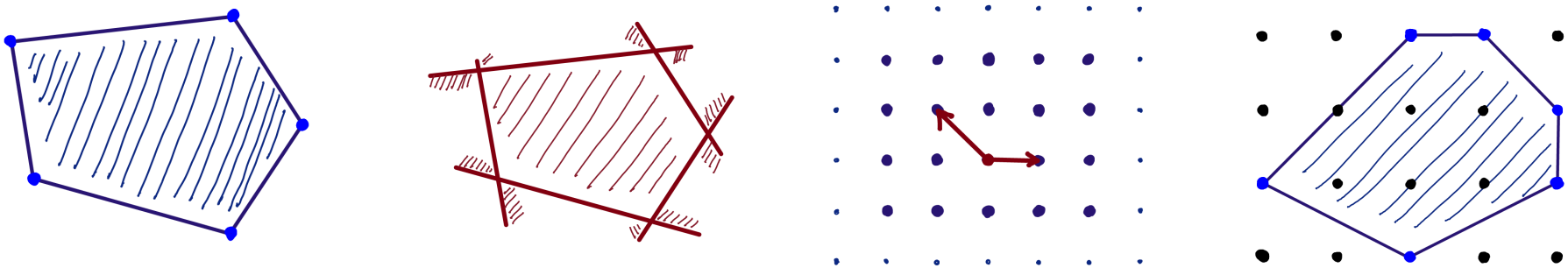

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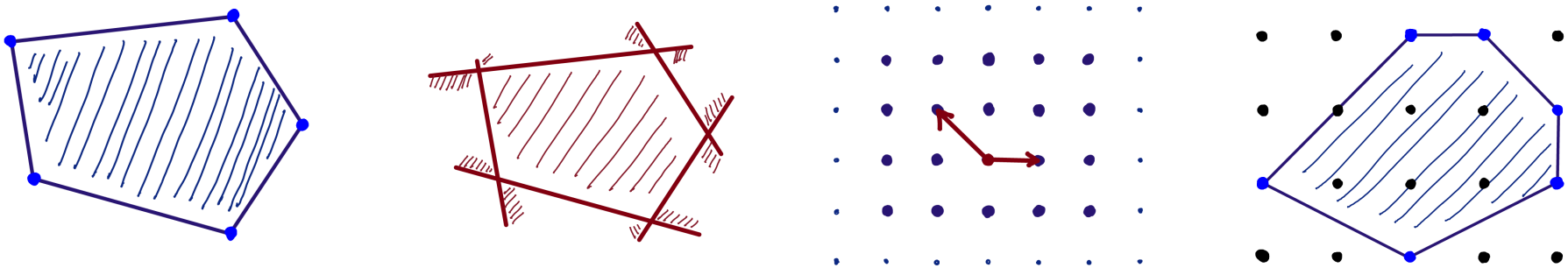

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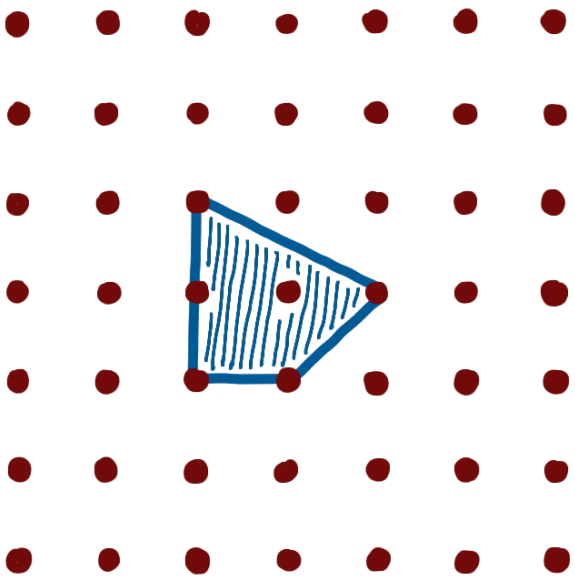

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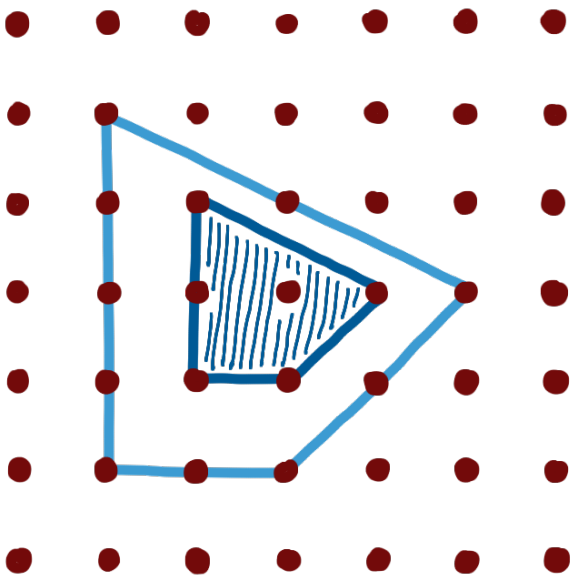

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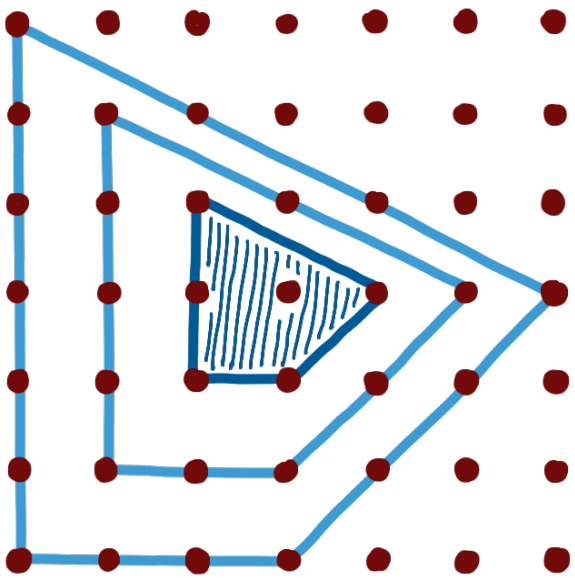

1, 6, 16, 31, . . .



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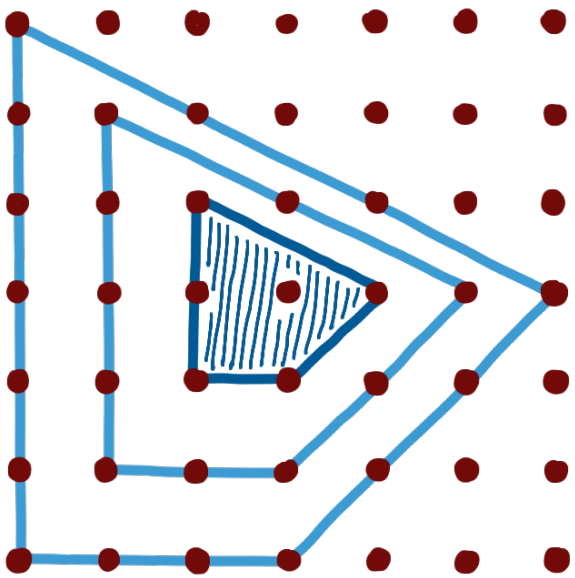

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for nonnegative integers $h^*_r, h^*_{r-1}, \ldots, h^*_0$

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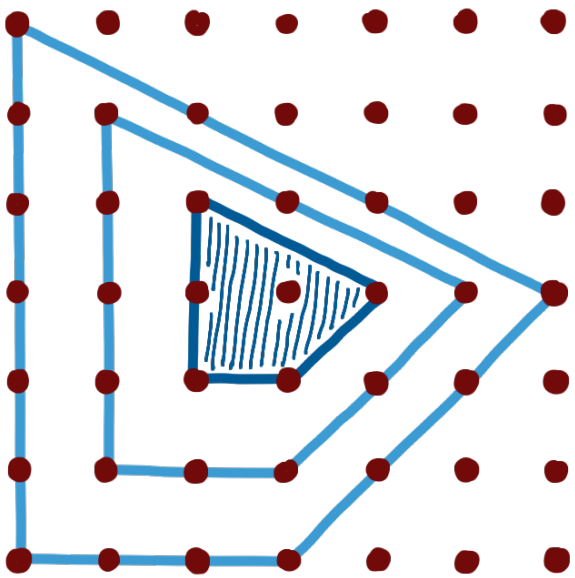

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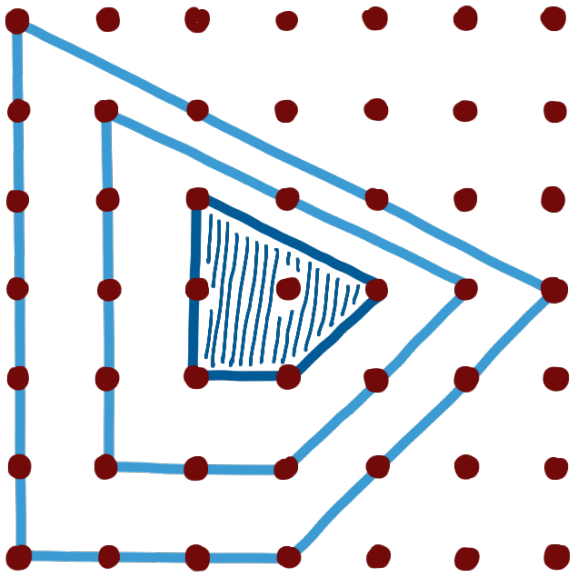

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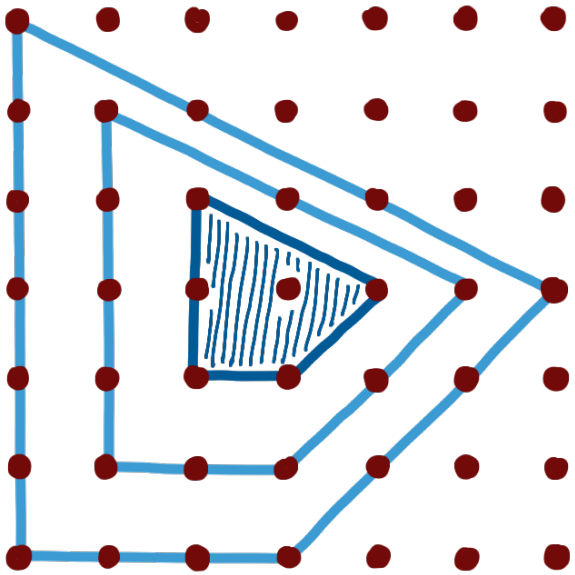

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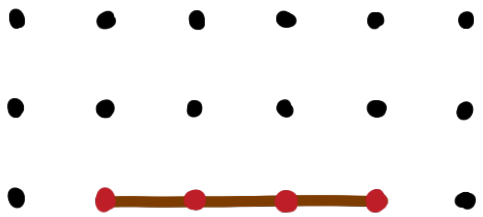

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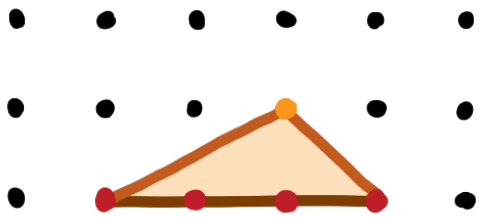

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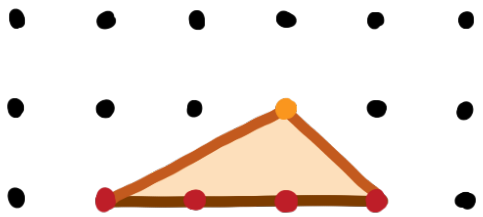

- $\deg P = 0$, then $P$ is a unit simplex
- lattice pyramids preserve the codegree
- classifications for codegree $\operatorname{codeg} = 1, 2$ (up to lattice pyramids) [Batyrev, Nill, Treutlein]



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# Cayley-Polytopes

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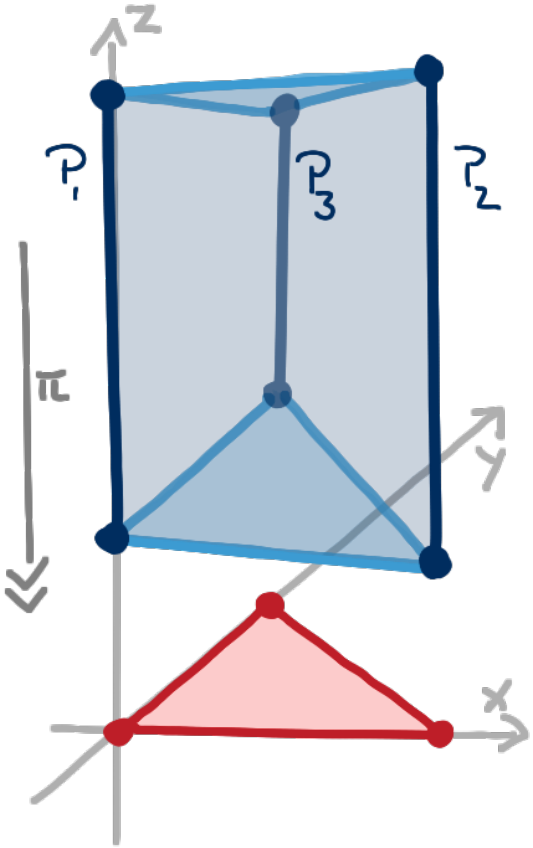
- Structure of polytopes with large codegree?

  $\longrightarrow$ polytopes with lattice projection onto unit simplex (:= Cayley polytopes)



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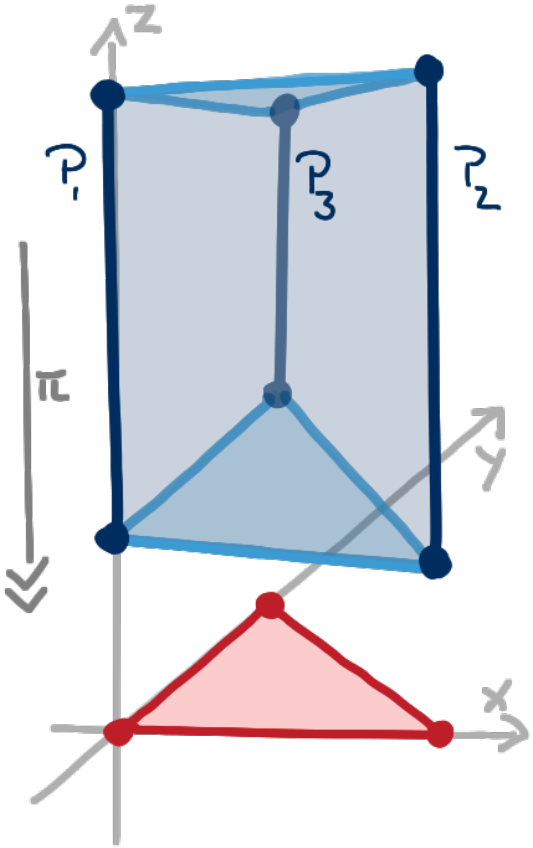




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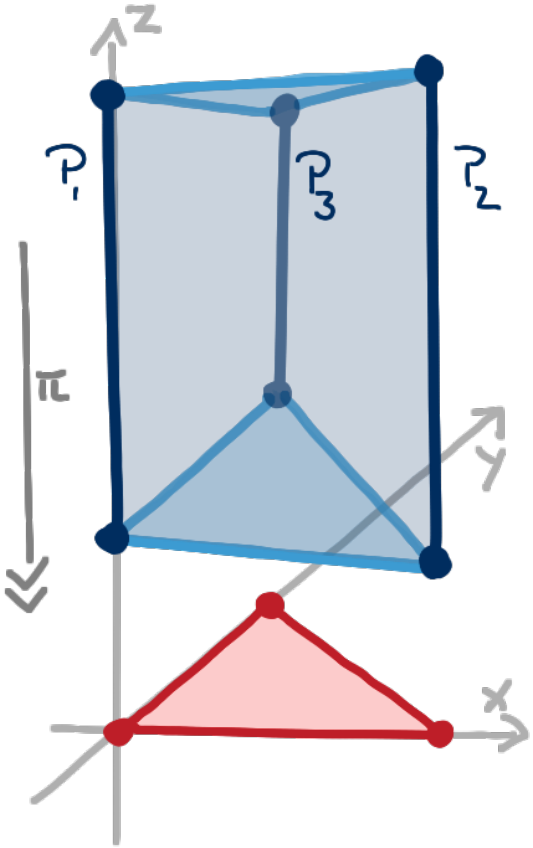
  ▷ true for *smooth* polytopes [Dickenstein, Nill, '10]

  (smooth := rays of each normal cone are lattice basis)



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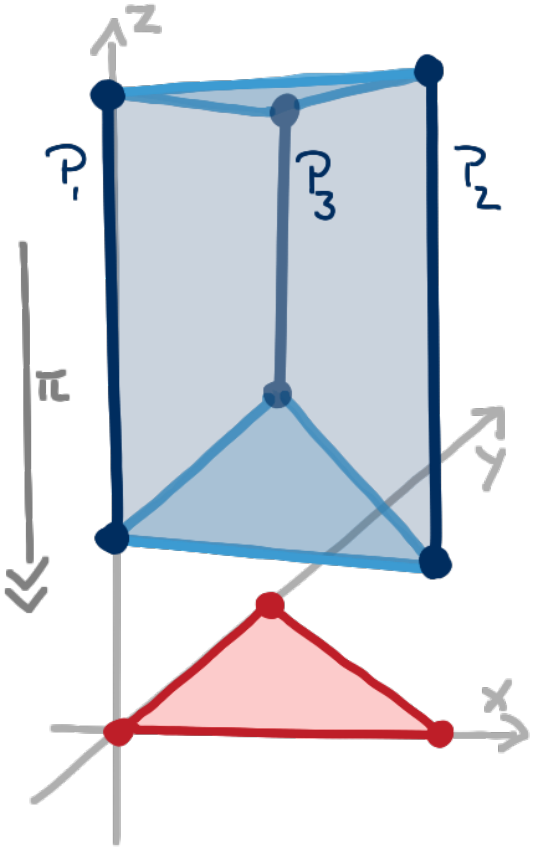




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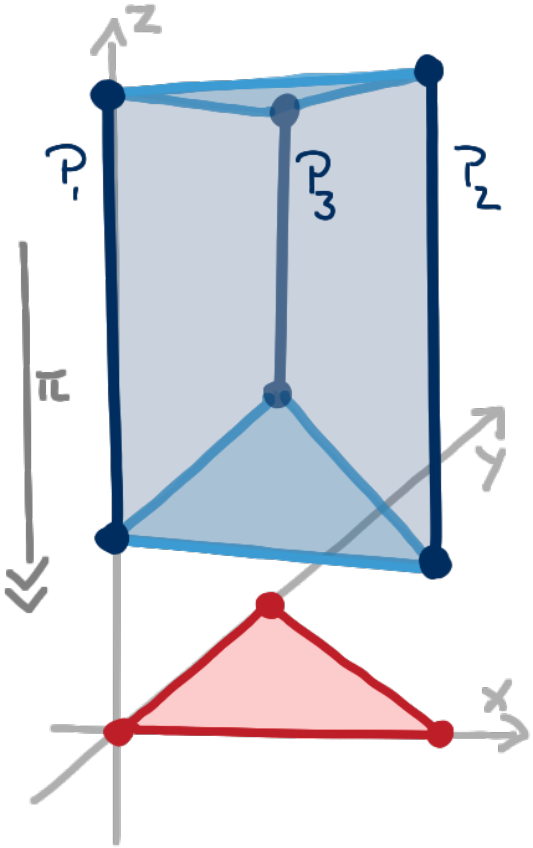




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# Adjunction and $\mathbb{Q}$-Codegree

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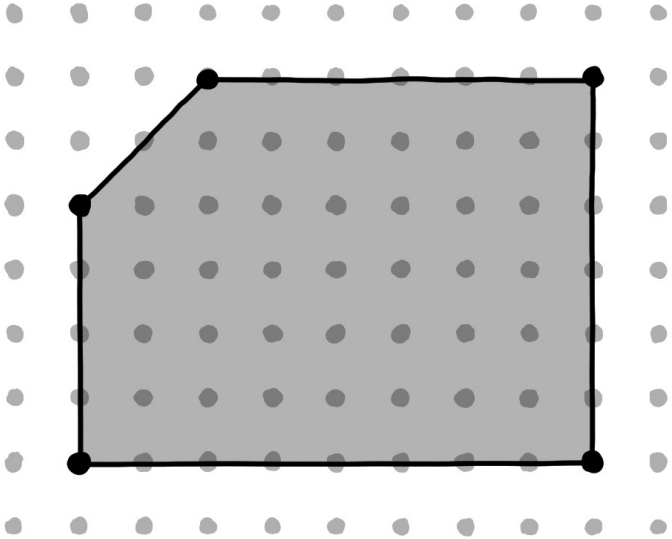



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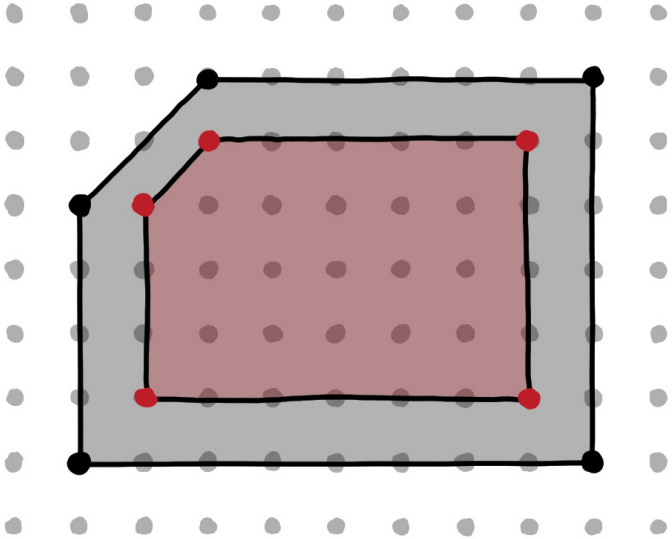

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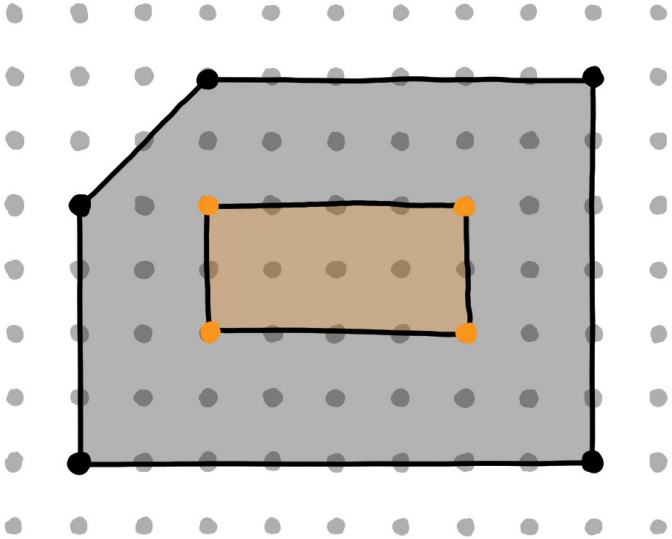

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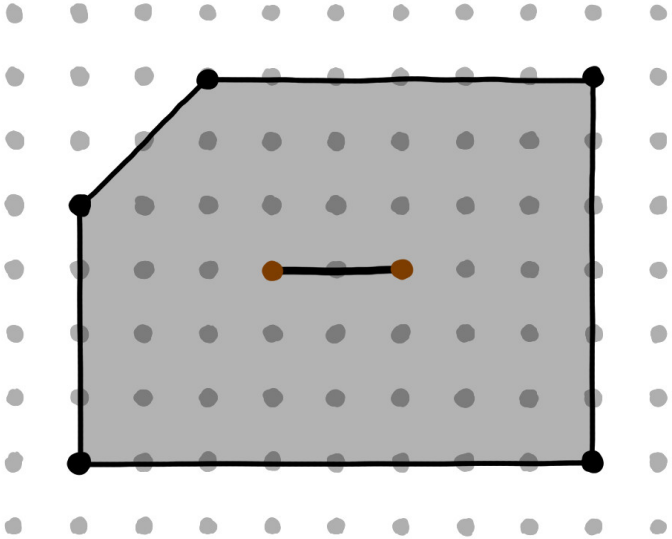

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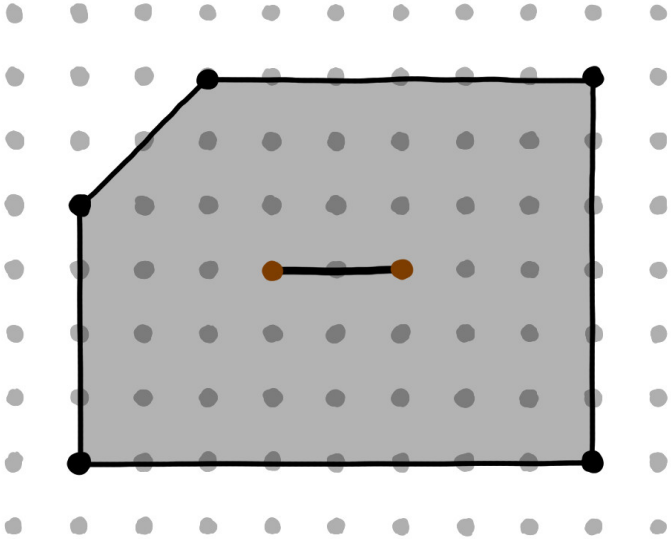



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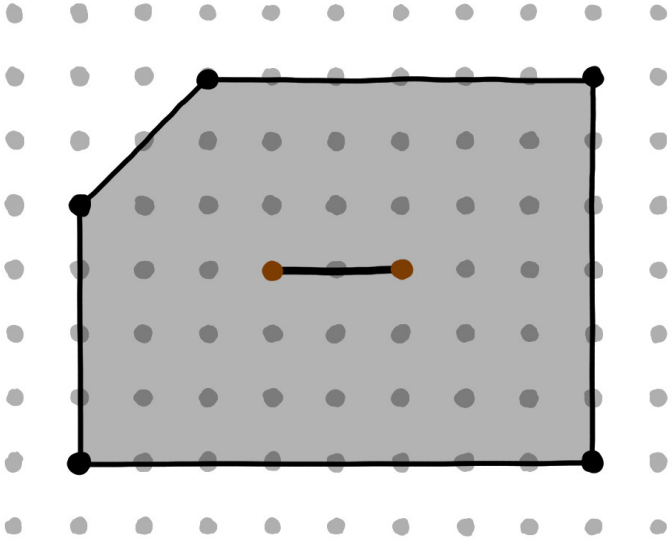

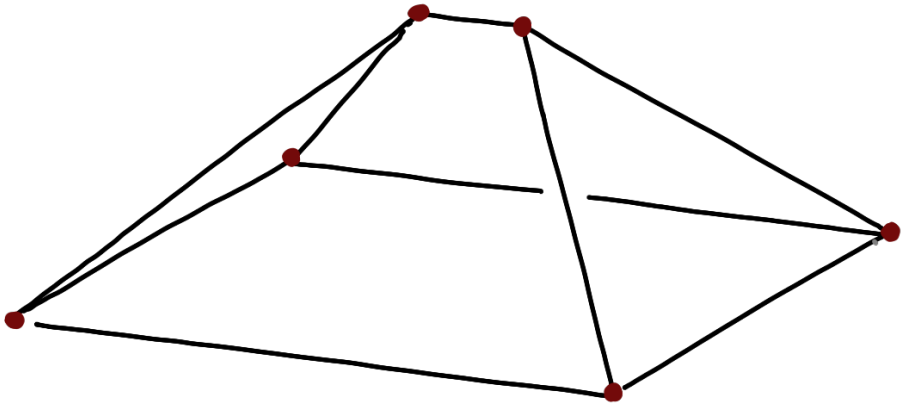

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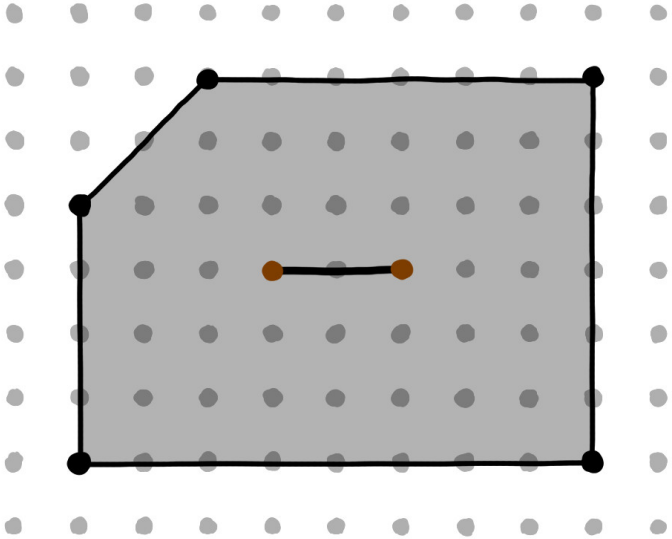

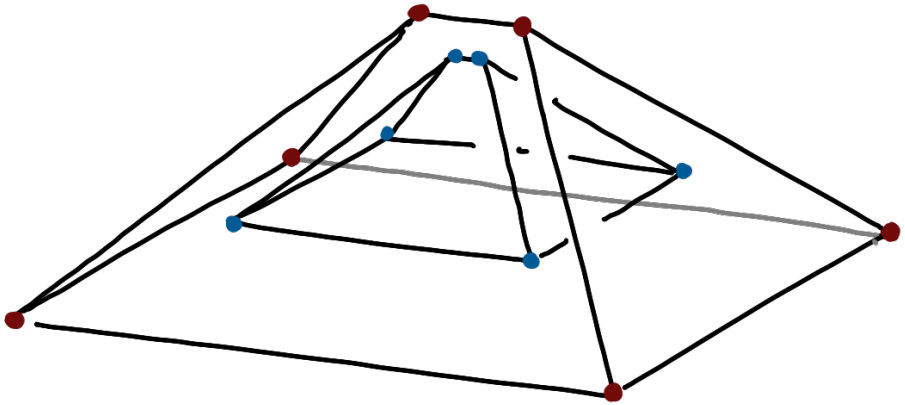



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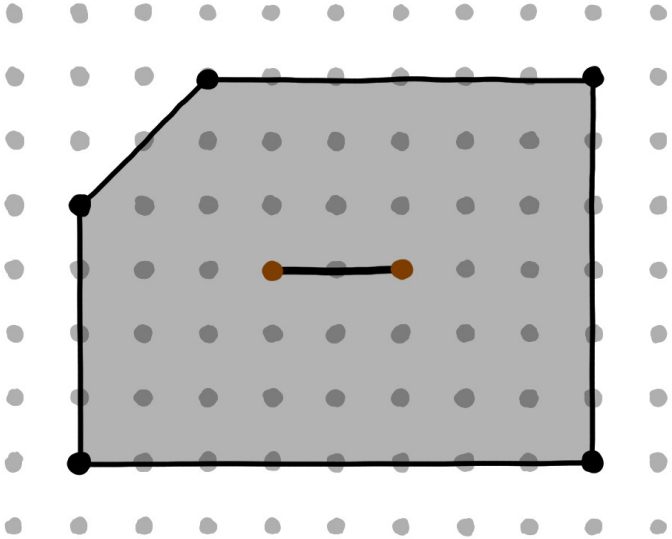

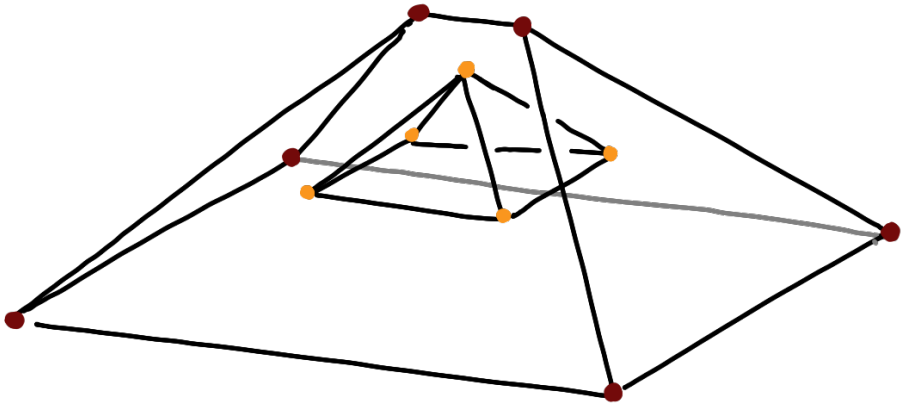



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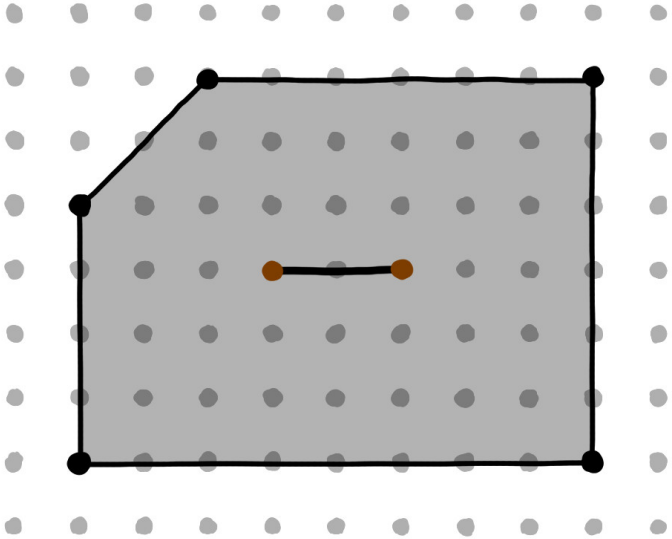

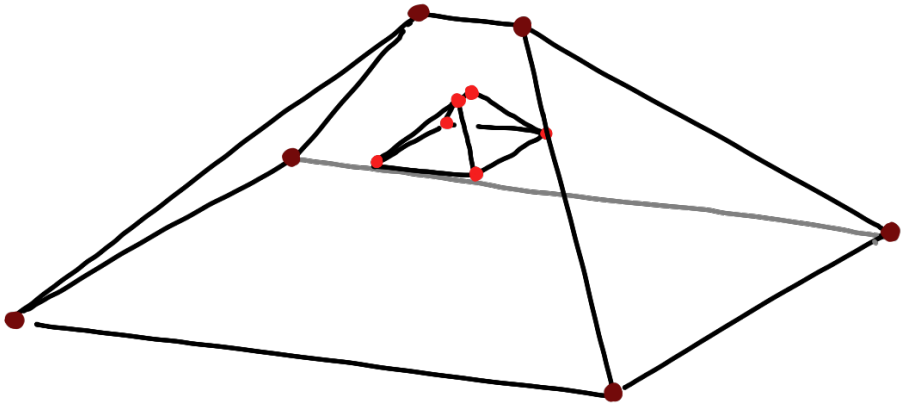



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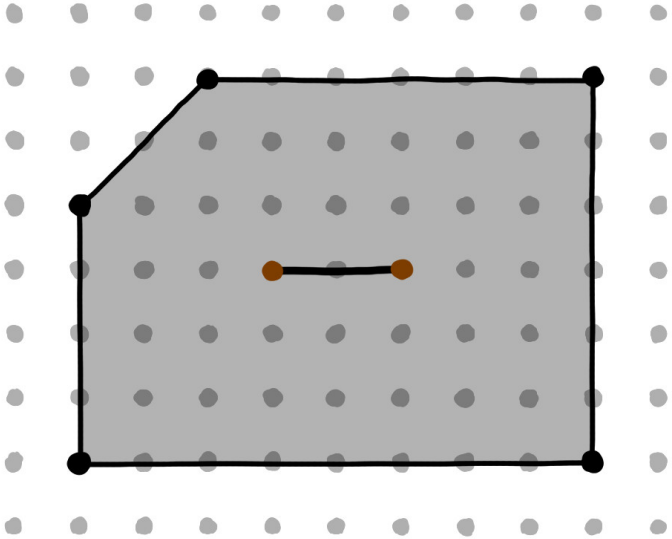

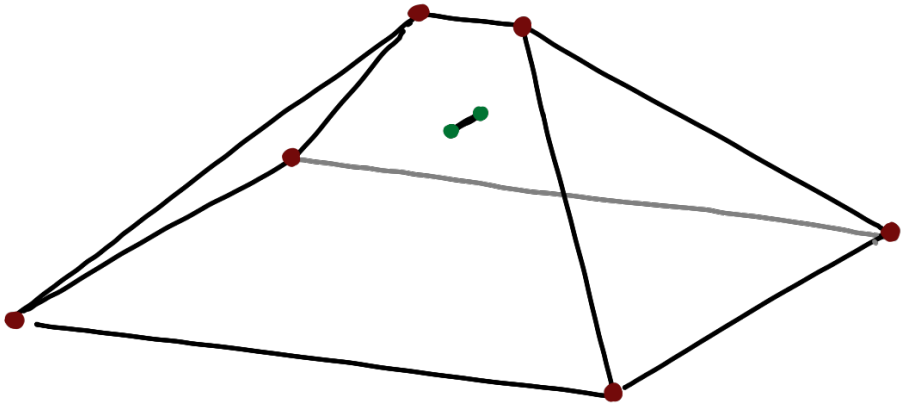



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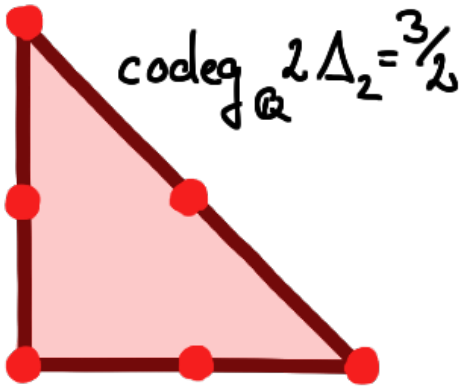




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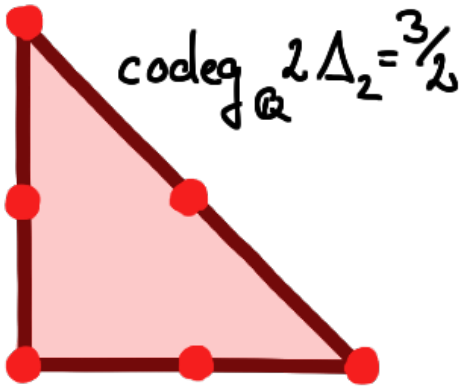


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- Conjecture: $\operatorname{codeg}_{\mathbb{Q}} P > \dfrac{d+1}{2}$ suffices



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# Toric Geometry

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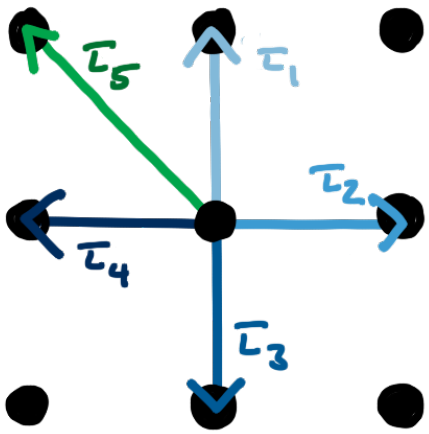




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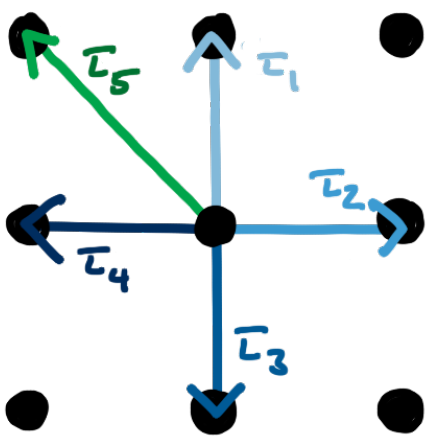




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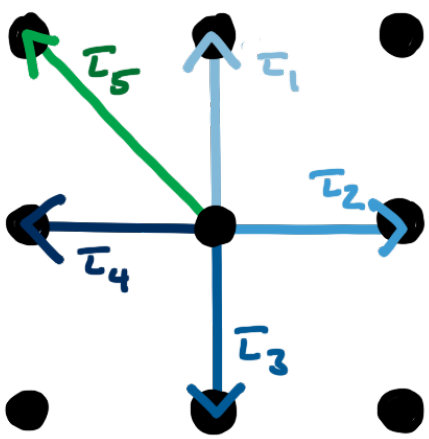




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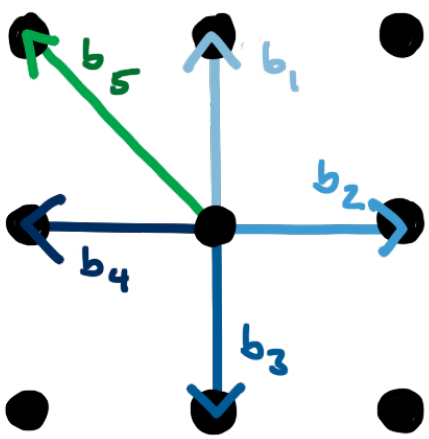


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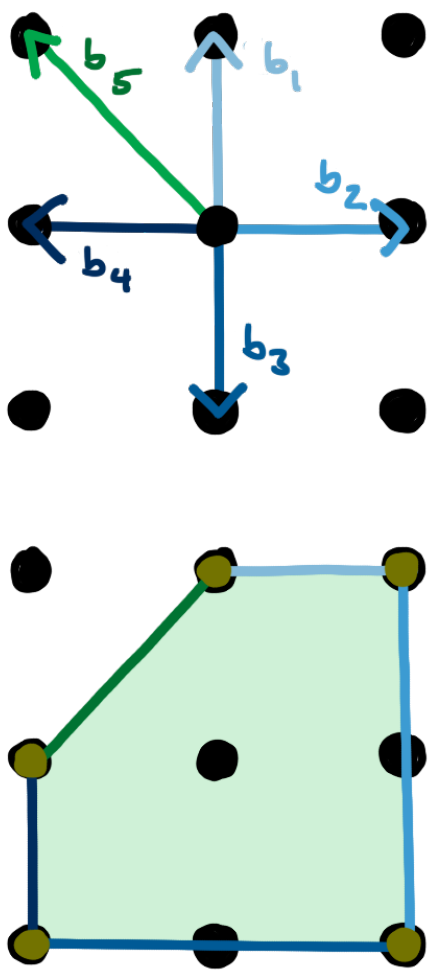




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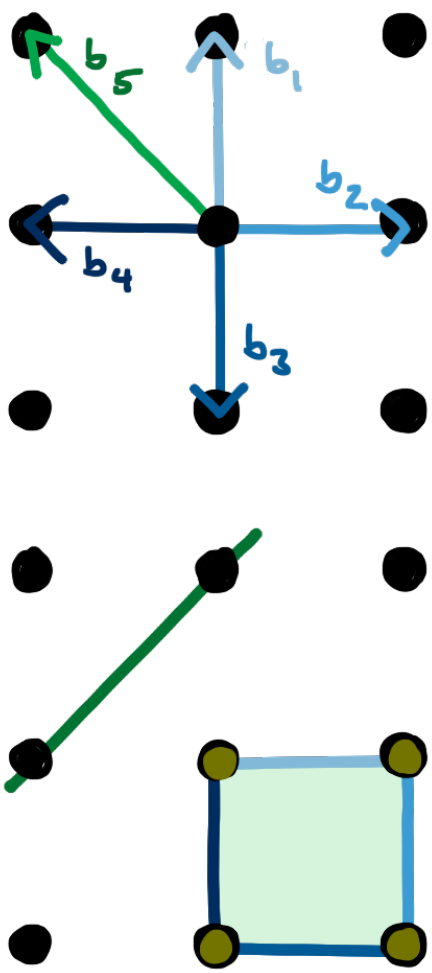




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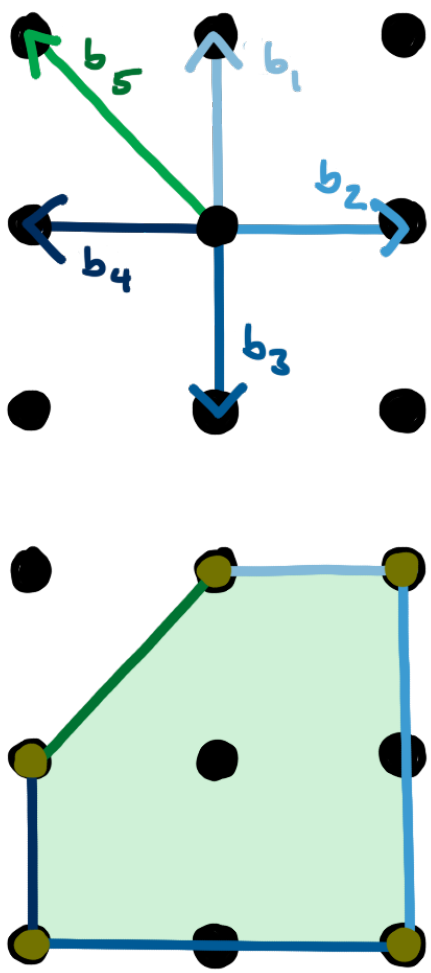




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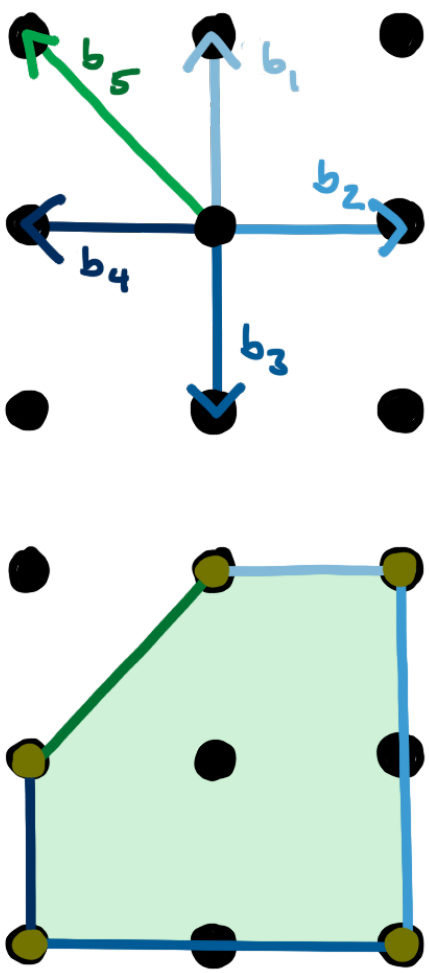
- $X$ is smooth (*i.e.* is a manifold):

minimal generators of every cone

are a lattice basis



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- family of adjoint divisors: $L + t \cdot K_X$
- (unnormalized) spectral value:

$$\sigma(X) \;:=\; \sup(t \mid L + t \cdot K_X \text{ is big})$$

[Sommese, Fujita]



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- family of adjoint divisors: $L + t \cdot K_X$
- (unnormalized) spectral value:

$$\sigma(X) \;:=\; \sup(t \mid L + t \cdot K_X \text{ is big})$$

[Sommese, Fujita]

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Theorem: $\sigma \geq {}^{(d+2)}/_{2}$ $\Longrightarrow$ morphism $\pi \;:\; \mathbb{P}(H_0 \oplus H_1 \oplus \cdots \oplus H_m) \;\longrightarrow\; X$
$H_i$ line bundles over toric variety of dim $\leq 2(n + 1 - \mu)$

[Di Rocco, Haase, Nill, P. '14]



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$$S_d := \left\{ \mu \;\middle|\; \begin{array}{l} \exists (X, L) \text{ smooth proj.} \\ d\text{-dim. pol. variety} \\ \text{with } \mu = \mu(L) \end{array} \right\}$$

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- first step essentially uses

  Theorem: $m, d \geq 1$
  Up to lattice equivalence, there are only finitely many lattice $d$-polytopes with $m$ interior lattice points.

[Hensley, Lagarias& Ziegler, Pikhurko]



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