Permutation Polytopes

Bielefeld, December 5, 2011

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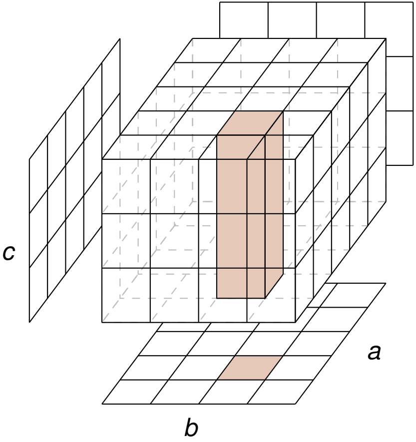

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Algorithmic Discrete Mathematics

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Permutation Polytopes: Examples

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- Cyclic groups: marginal polytopes

Group $\longleftrightarrow$ Polytope

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- $\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$G_1 := \langle\ (1\ 2), (3\ 4)\ \rangle \qquad P(G_1) = C_2 \in \mathbb{R}^{16} \qquad \text{(square)}$$

$$G_1 := \langle\ (1\ 2), (3\ 4)(5\ 6)\ \rangle \qquad P(G_2) = C_2 \in \mathbb{R}^{36}, \quad P(G_2) \cong P(G_1)$$

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ρ_1, ρ_2 representations of G with same irred. factors $\Longrightarrow$ $P(\rho_1), P(\rho_2)$ affinely equivalent

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    $\rightarrow$ allow group isomorphism $\phi : \rho_1 \rightarrow \rho_2$
- Question: Does this suffice if
  - G_1, G_2 abstractly isomorphic
  - $P(G_1), P(G_2)$ aff. isomorphic ?

Combinatorics

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- $g \in G$, $g = z_1 \cdot z_2 \cdots z_k$ disjoint cycles
- $h \in G$, h sub-element of g if $h = z_{i_1} \cdot z_{i_2} \cdots z_{i_j}$ for $\{i_1, \ldots, i_j\} \subseteq [k]$

[Guralnick& Perkinson]

▷ example: $g = (1\ 2)\,(3\ 4\ 5)\,(6\ 7)$ $h = (1\ 2)\,(6\ 7)$

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Theorem [Guralnick& Perkinson]

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Theorem [Guralnick& Perkinson]

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- F_g centrally symmetric
- G transitive $\Rightarrow \operatorname{diam}(P(G)) \leqslant 2$.

Combinatorics

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$P(G)$ combinatorial product $\iff$ G product of permutation groups

Then: $P(G) = P_1 \times P_2, \quad G = G_1 \times G_2, \quad P_i = P(G_i)$.

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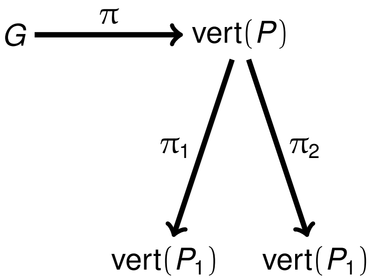

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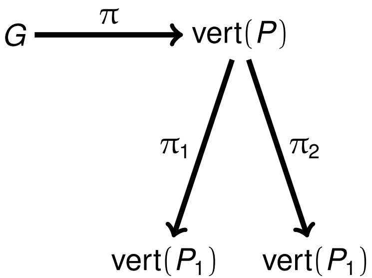

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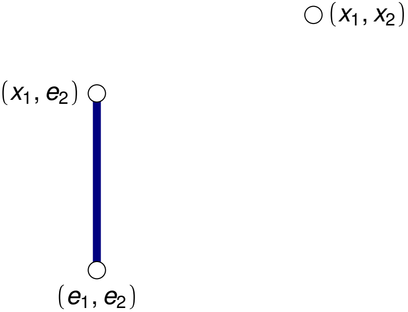

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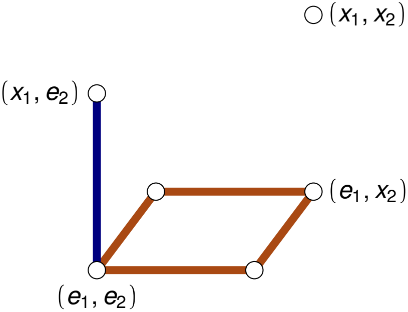

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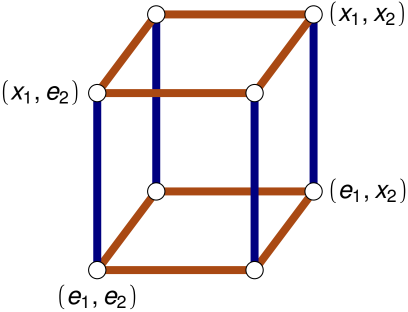

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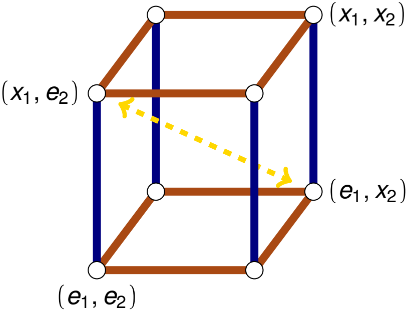

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Then: $P(G) = P_1 \times P_2, \quad G = G_1 \times G_2, \quad P_i = P(G_i).$

sketch of proof:

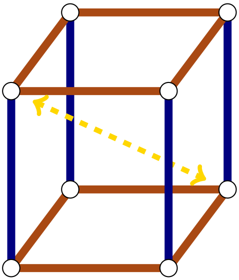
- "$\Leftarrow$": matrices are block diagonal
- "$\Rightarrow$": $\pi : G \to \mathrm{vert}(P(G))$,

  $\pi_i : \mathrm{vert}(P) \to \mathrm{vert}(P_i)$.

  $\pi(e) = (e_1, e_2) \in \mathrm{vert}(P)$.

  Let $G_{2-i} := \pi^{-1}(\pi_i^{-1}(e_i))$

  - $\mathrm{supp}(G_1) \cap \mathrm{supp}(G_2) = \varnothing$.
  - G_1, G_2 are subgroups of G.



Low dimensional Classification

- use classification of 0/1-polytopes

Low dimensional Classification

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- use classification of $0/1$-polytopes
- types of permutation polytopes up to dimension 4:
  - Δ_2, C_2,
  - Δ_3, C_3, $\text{prism}(\Delta_2)$,
  - Δ_4, C_4, $\text{prism}(\Delta_3)$, $\Delta_2 \times \Delta_2$, $C_2 \times \Delta_2$, cross_4, B_3

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- combinatorial types of faces up to dimension 4
  - dim 2: 2, dim 3: 5, dim 4: 19 – 21
- G group with representation ρ and $\dim(P(\rho)) = n$.

  Question: Does G have a stably equivalent representation in S_{2n}?

  (known: any combinatorial type of k-face of B_n appears in B_{2k})

Cyclic Permutation Polytopes

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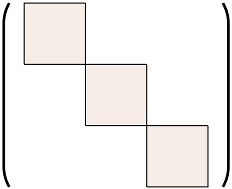
- $g = z_1 \cdot z_2 \cdots z_k$ pairwise disjoint cycles z_i of length t_i

  $\longrightarrow$ permutation matrix is block diagonal



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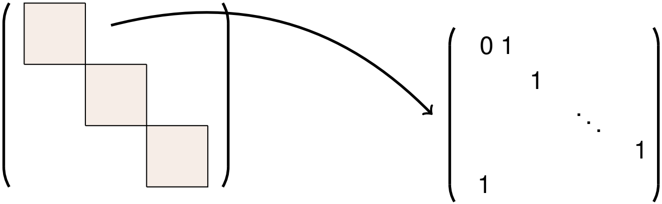

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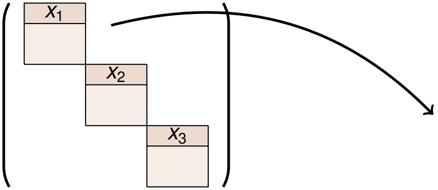

$$\begin{pmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & \end{pmatrix}$$

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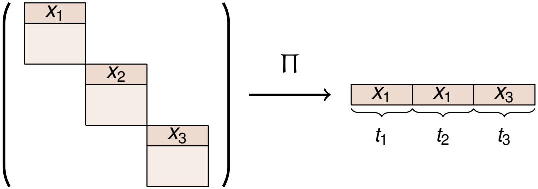

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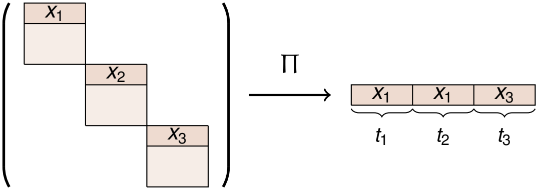

- $t := \sum t_i$
- isomorphism $\Pi : \mathbb{R}^{n \cdot n} \longrightarrow \mathbb{R}^t$
- $v = g^m \longrightarrow x_j$ is the $(m \mod t_j)$-th unit vector in $\mathbb{R}^{t_j}$

Cyclic Permutation Polytopes: Faces

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- $g = z_1 \cdot z_2 \cdots z_k$ pairwise disjoint cycles z_i of length t_i

- low-dimensional faces:
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Proposition complete graph

$$\longleftrightarrow$$

for all $I \subseteq [k]$: $d_I = d$ or $d_{I^c} = d$.

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Algorithmic Discrete Mathematics

Cyclic Permutation Polytopes: Faces

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Two Cycles

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Theorem $P(G) =$ r–fold join of $\Delta_{t_1/r\,-\,1} \times \Delta_{t_2/r\,-\,1}$

- $r := \gcd(t_1, t_2)$

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- $r := \gcd(t_1, t_2)$

example: $g = (1\ 2\ 3)(4\ 5)$:

$$\begin{array}{llll} g^0: & (1,0,0,\ 1,0) & g^1: & (0,1,0,\ 0,1) \\ g^2: & (0,0,1,\ 1,0) & g^3: & (1,0,0,\ 0,1) \\ g^4: & (0,1,0,\ 1,0) & g^5: & (0,0,1,\ 0,1) \end{array}$$

Three Cycles

- $G = \langle\, z_1 \cdot z_2 \cdot z_3 \,\rangle$ disjoint cycles of length t_1, t_2, and t_3.

Three Cycles

TECHNISCHE UNIVERSITÄT DARMSTADT

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$$(t_1, t_2, t_3) = (6, 10, 15): \quad \text{30 vertices in dimension 21,} \quad \text{211 facets}$$

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Three Cycles

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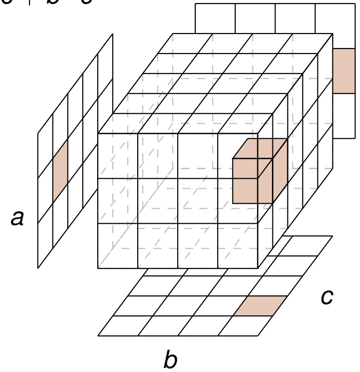

(a,b,c)=(3,4,5), $m = 31$:

$$\begin{array}{ccccccc} 31 & \triangleq & (1,3,1) & \triangleq & ((1,3),\ (1,1),\ (3,1)) & \triangleq & (\mathbf{e}_7,\ \mathbf{e}_1,\ \mathbf{e}_{11}) \\ \in [abc] & & \in [a] \times [b] \times [c] & & \in [ab] \times [ac] \times [bc] & & \in \mathbb{R}^t \end{array}$$

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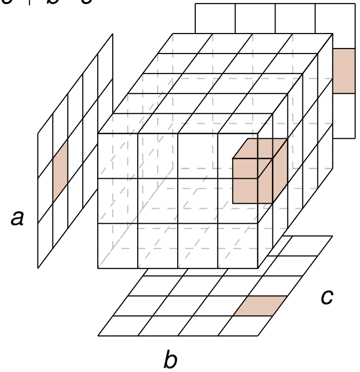

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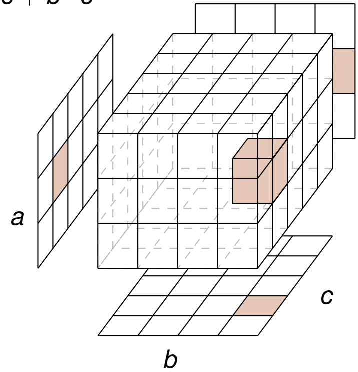

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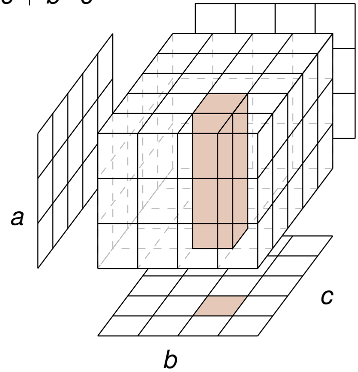

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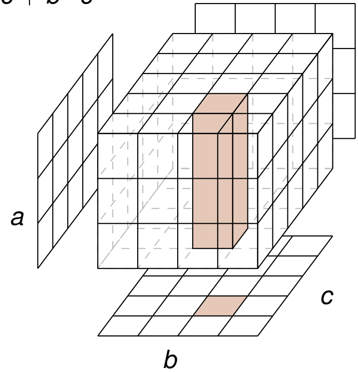

- marginal map: $\quad M : \mathbb{R}^{abc} \longrightarrow \mathbb{R}^t$
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- M given by matrix:
  $P(a, b, c) = \mathrm{conv}(\mathrm{columns}(M))$ $\qquad \rightarrow$ marginal polytope

Marginal Polytopes

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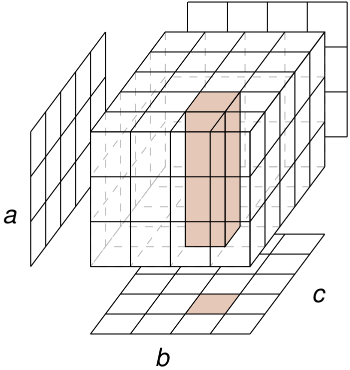
  $\rightarrow$ linear/integer optimization over $P(M)$.



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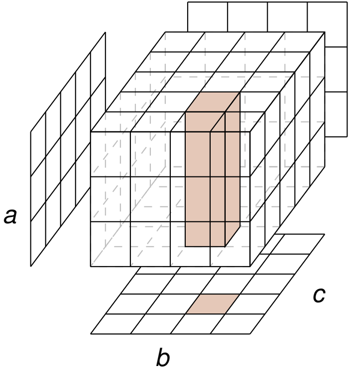

- Faces/Facets studied via cut polytopes

  [Deza&Laurent, Sullivant, Sontag&Jaakkola, . . .]

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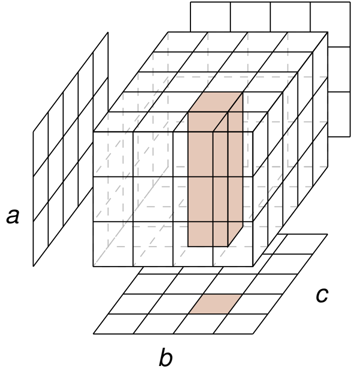

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  - for $P(2, 3, 5)$, $P(3, 4, 5)$ see e.g.

    http://markov-bases.de/ [Kahle&Rauh]

Faces

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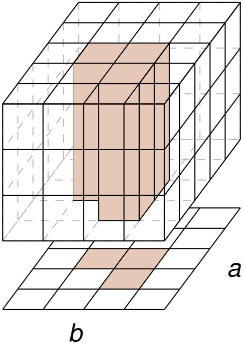

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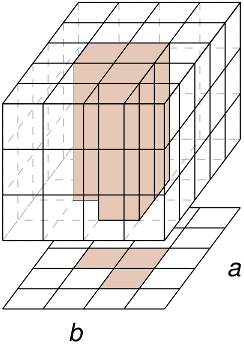

Face Lemma

$S_{a,b} \subsetneq [ab]$, $S_{a,c} \subsetneq [ac]$, $S_{b,c} \subsetneq [bc]$

+ some conditions

Then $F_{a,b} \cup F_{a,c} \cup F_{b,c}$ is a face of $P(a,b,c)$.

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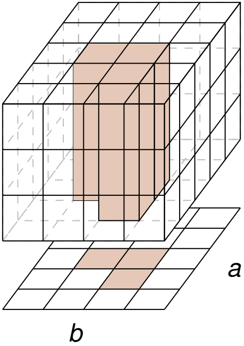

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Corollary

$P(a,b,c)$ is 3-neighborly (cf. [Kahle '09])

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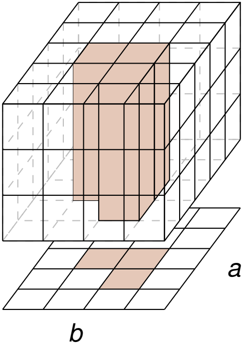

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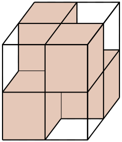
but never 4-neighborly



Facets: Checkerboard

- $\varnothing \subsetneq I_j \subsetneq [j]$ for $j \in \{a, b, c\}$. $\bar{I}_j := [j] \setminus I_j$
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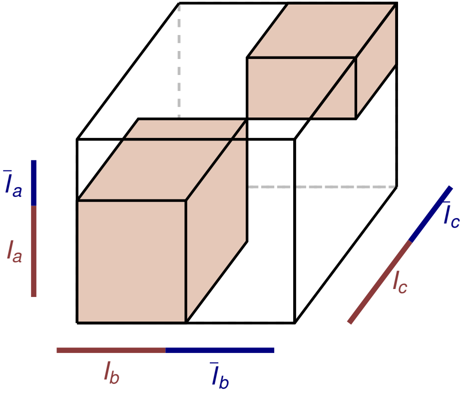

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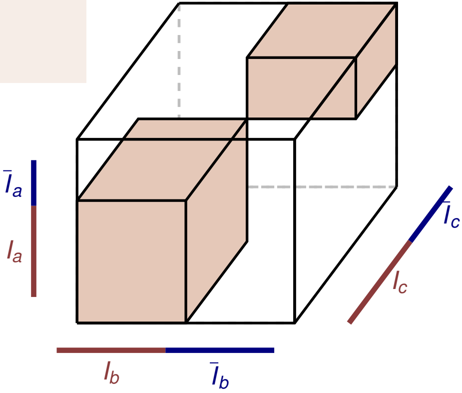

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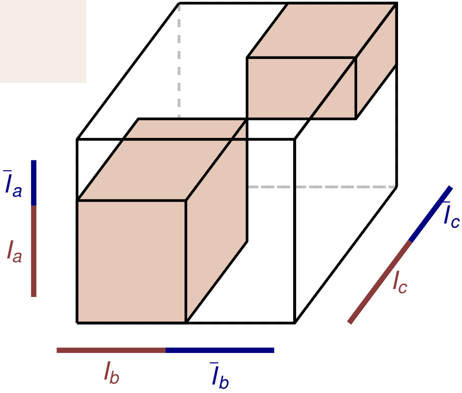

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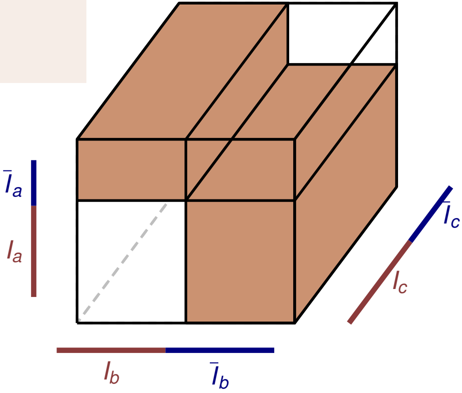

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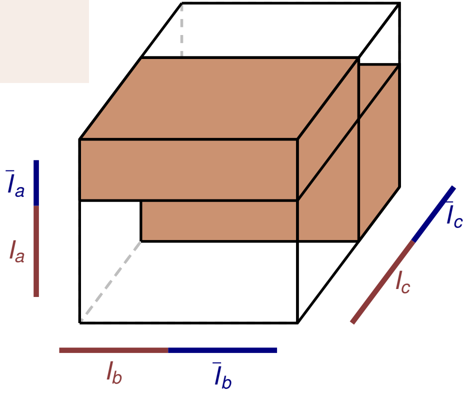

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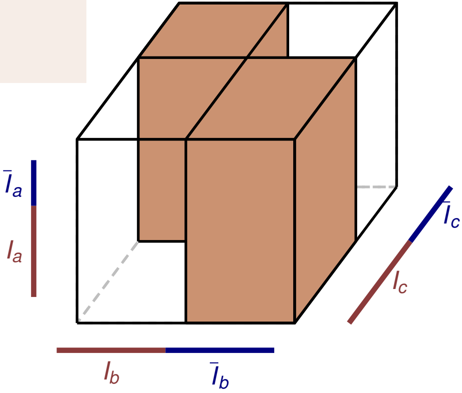

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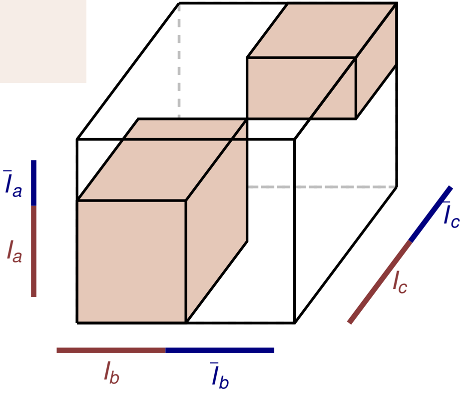

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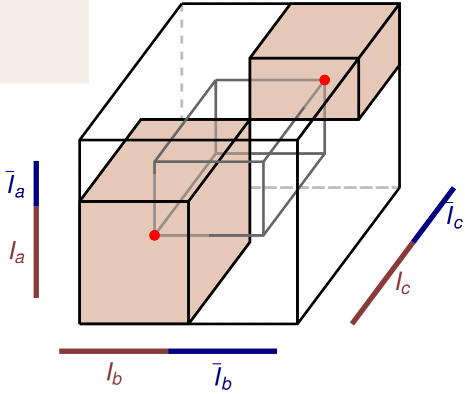

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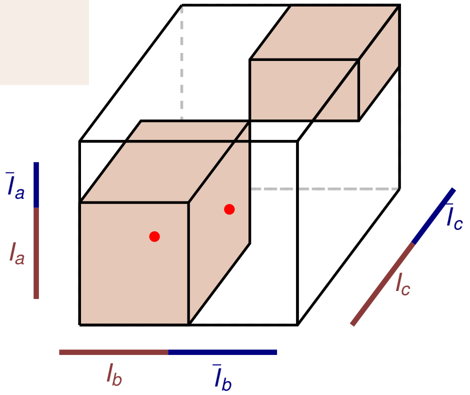

Facets: Permutations

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  proper partition $(A_1, A_2, \ldots, A_q)$, of $[a]$ into q sets, etc.

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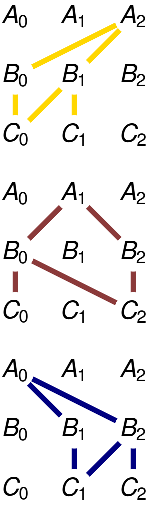

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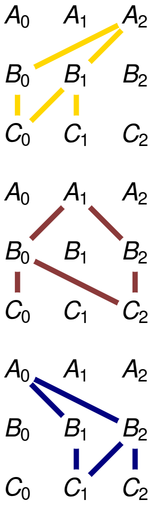

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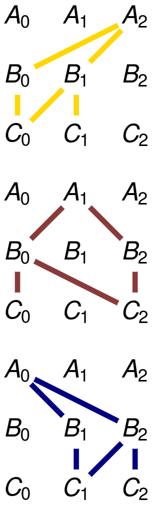
Conjecture: For $a = 2$ this suffices.



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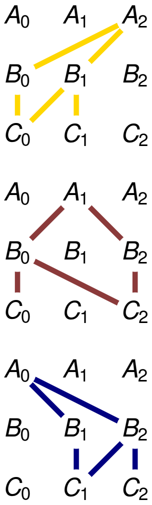

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- $M(a, b, c)$: marginal polytope for 3-way table with sides a, b, c

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Further Questions

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- cyclic polytopes with three orbits: do we have all facets?
- cyclic polytopes with more orbits?
- P d-dimensional permutation polytope.
  - Is there $P(G)$ affinely equivalent with $G \leqslant S_{2d}$
  - known: any combinatorial type of k-face of B_n appears in B_{2k}.
- size of $\mathrm{Aut}(P(G))$?
  - known for cyclic groups [Rehn 2010]
- ...