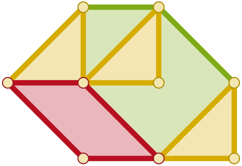
Ehrhart Theory

Polyhedra



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- ▶ Minkowski sum $\quad X + Y \ = \ \{x + y \ \mid \ x \in X, \ y \in Y\}$

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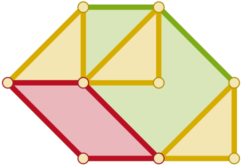

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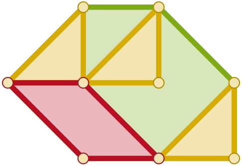

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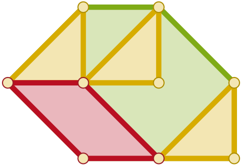

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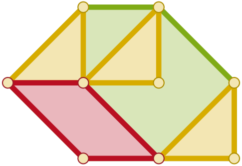

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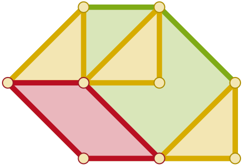

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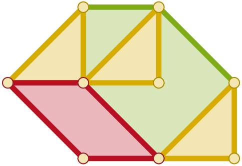

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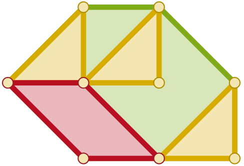

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$$:\Longleftrightarrow \quad \{\sum \lambda_{\mathbf{x}} \mathbf{x} \mid 0 \leq \lambda_{\mathbf{x}} \leq 1, \sum \lambda_{\mathbf{x}} = 1\} =: \mathrm{conv}(X)$$

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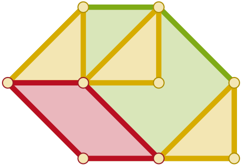

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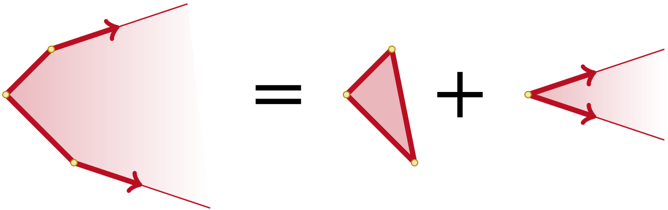
- polyhedron $P = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \text{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_r) + \text{cone}(\mathbf{w}_1, \ldots, \mathbf{w}_s) \subseteq \mathbb{Q}^d$



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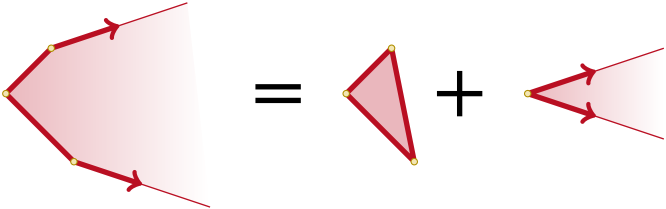

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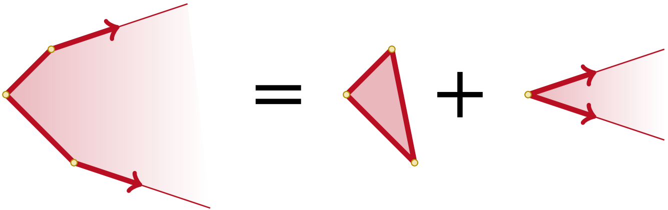

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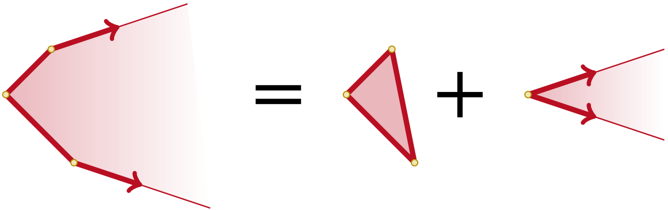

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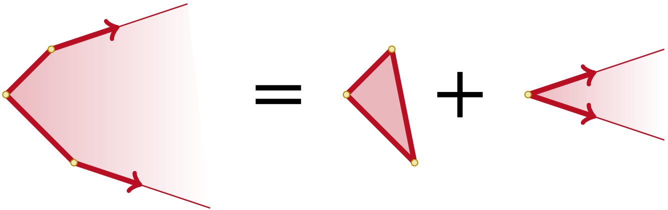

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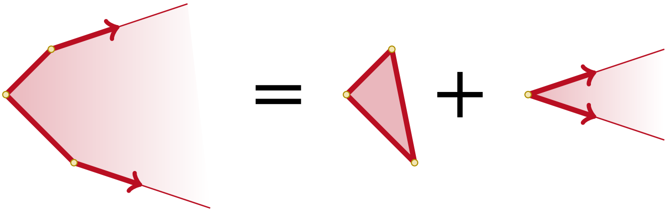

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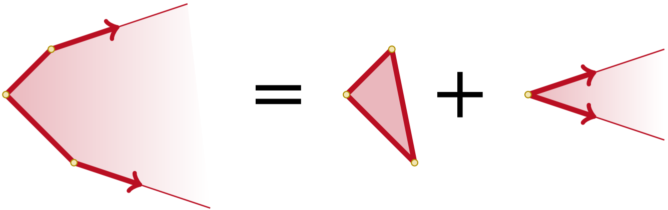

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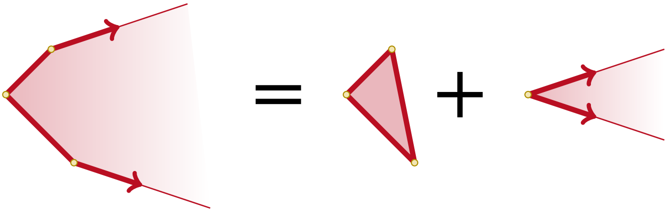

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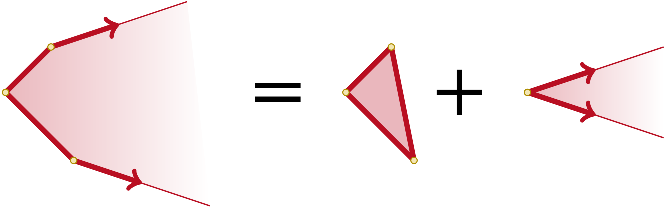

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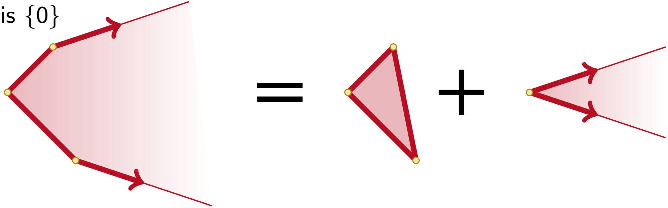
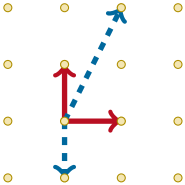


Lattice Polytopes

- $\Lambda \subseteq \mathbb{R}^d$ lattice $\quad :\Longleftrightarrow \quad$ Λ is discrete, additive, abelian group



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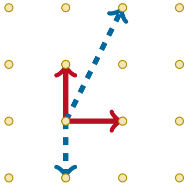

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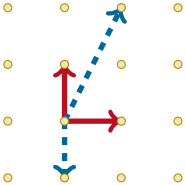

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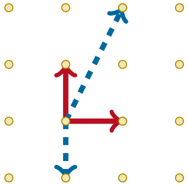
- fundamental parallelepiped $\Pi := \{\sum \lambda_i \mathbf{a}_i \mid 0 \leq \lambda_i < 1\}$
  - $\mathbf{x} \in \mathbb{R}^d$ represented uniquely as $\mathbf{x} = \mathbf{y} + \mathbf{z}$, for $\mathbf{y} \in \Pi$ and $\mathbf{z} \in \Lambda$



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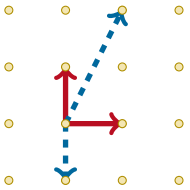

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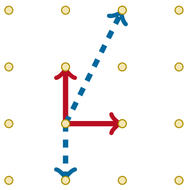

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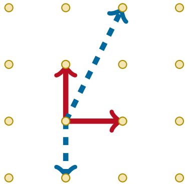
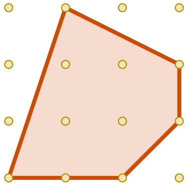
- $\mathbf{w}_i \in \mathbb{Z}^d$ primitive $:\Longleftrightarrow$ $\lambda \mathbf{w}_i \notin \mathbb{Z}^d$ for all $0 < \lambda < 1$
- lattice polytope $:\Longleftrightarrow$ polytope $P \subseteq \mathbb{R}^d$ with vert $P \subseteq \mathbb{Z}^d$.





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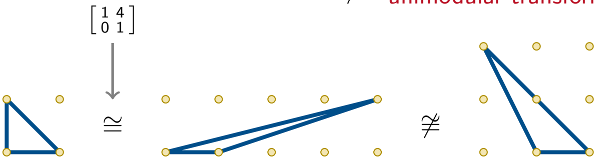
  $\longrightarrow$ unimodular transformations



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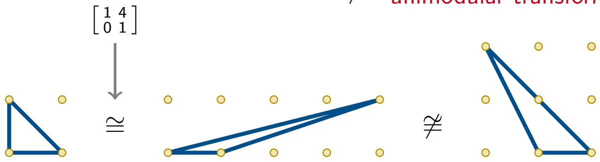

- Proposition: unimodular transformations preserve #(lattice points) and volume

  (they do *not* preserve lengths and angles)

Ex

Why Count Lattice Points

- knapsack type problems
  - $\mathbf{v}$ cost or volume, $\mathbf{s}$ size, solve
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$\longrightarrow$ lattice points in the polytope defined by

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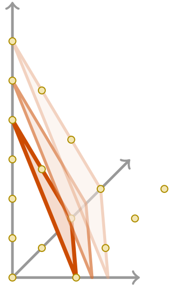
$$k = x_1\cdot 2 + x_2\cdot 1 + x_3\cdot \tfrac{1}{2}$$



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$\longrightarrow$ 714,574,663,432 tables.

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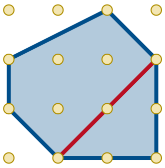

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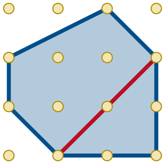

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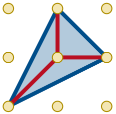

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- ▶ further observations:
  - ▷ leading coefficient is volume
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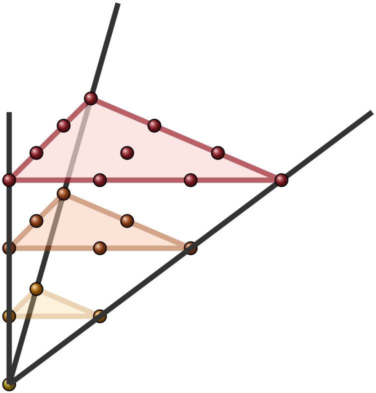

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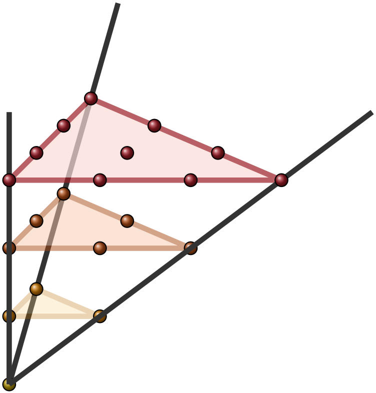

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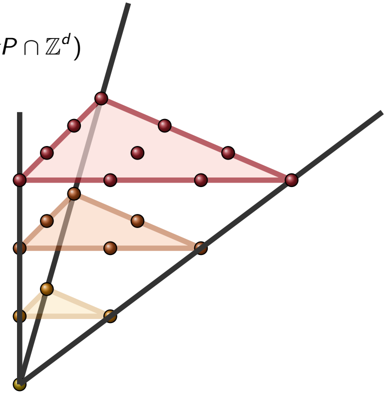

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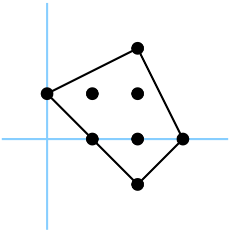
- example

$$P = \operatorname{conv} \begin{bmatrix} 0 & 2 & 2 & 3 \\ 1 & -1 & 2 & 0 \end{bmatrix}$$



$$\hat{G}_P := \sum_{a \in P \cap \mathbb{Z}^d} t^a = t_1^2 t_2^2 + t_2 + t_1 t_2 + t_1^2 t_2 + t_1 + t_1^2 + t_1^3 + t_1^2 t_2^{-1} \in \hat{L}$$

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Example: $\triangleright$ $P = [0, \infty) \longrightarrow \hat{\mathrm{G}}(t) = 1 + t + t^2 + \ldots = \sum_{a \in \mathbb{Z}_{\geq 0}} t^a$

$g(t) := (1 - t) \longrightarrow g\hat{\mathrm{G}}(t) = 1$, so $\hat{G}$ is summable

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- Def: P polyhedron, $\hat{G}_P(t)$ summable.

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▶ Def: $\Pi(\mathbf{v}_1, \ldots, \mathbf{v}_d) := \{\mathbf{x} = \sum \lambda_i \mathbf{v}_i \mid 0 \leq \lambda_i < 1\}$

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$\longrightarrow$ we have encoded lattice points in a simplicial cone!

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- ▷ $S \;:=\; \mathrm{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_d)$ simplex
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  - ▷ $\mathrm{ht}(\overline{\mathbf{w}}) \;:=\; w_0$ height of $\overline{\mathbf{w}}$ in $\overline{\Pi}$ $\longrightarrow$ at most d

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- ▷ $S \; := \; \mathrm{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_d)$ simplex
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    $\longrightarrow$ non-negative and integral

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- Theorem (Ehrhart's Theorem for Simplices)

  S simplex, then

  $$\mathrm{Ehr}_S(t) \;=\; \sum_{k\geq 0} \mathrm{ehr}_S(k)t^k \;=\; \frac{h^\star(t)}{(1-t)^{d+1}}$$

  - $h^\star$ polynomial of degree $\leq d$, $h^\star(0) \neq 0$, non-negative integer coefficients
  - $\mathrm{ehr}_S(z)$ polynomial of degree d **Ex**
  - $\mathrm{ehr}_S(z) \;=\; \binom{z+d}{d}h_0^\star + \binom{z+d-1}{d}h_1^\star + \cdots + \binom{z+1}{d}h_{d-1}^\star + \binom{z}{d}h_d^\star$ **Ex**

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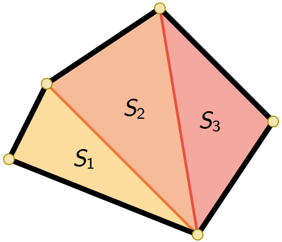

Ehrhart's Theorem

- how do we extend this to general polytopes? $\longrightarrow$ triangulate!
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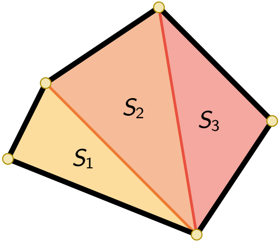

$$\begin{aligned}\mathrm{ehr}_P(k) \;=\; & \mathrm{ehr}_{S_1}(k) + \mathrm{ehr}_{S_2}(k) + \mathrm{ehr}_{S_3}(k) \\ & - \mathrm{ehr}_{S_1\cap S_2}(k) - \mathrm{ehr}_{S_2\cap S_3}(k) \\ & + \mathrm{ehr}_{S_1\cap S_2\cap S_3}(k)\end{aligned}$$

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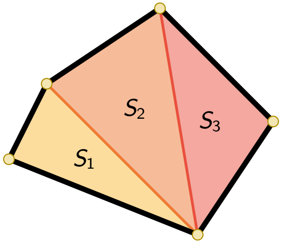

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- Theorem (Ehrhart's Theorem)

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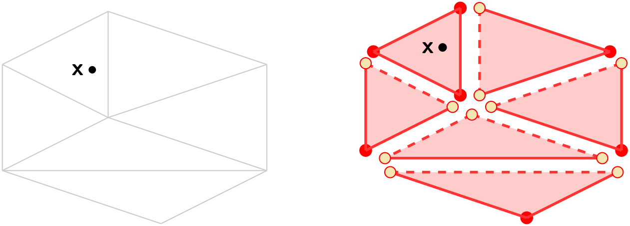

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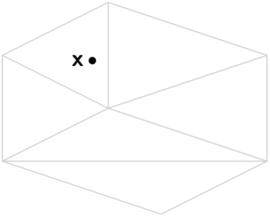

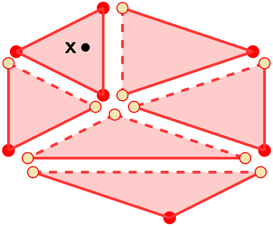

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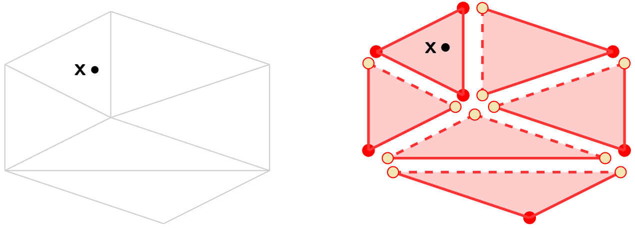

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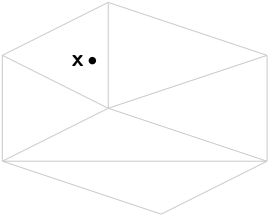

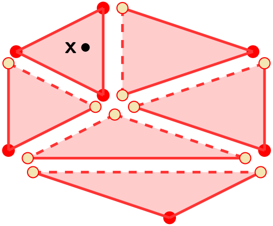

  - ▷ lattice points in intersections only counted once
  - ▷ preserves non-negativity
  - ▷ need to define this for the cones obtained from such a triangulation

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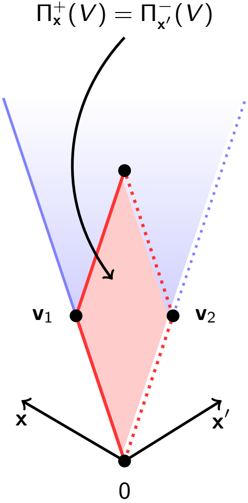

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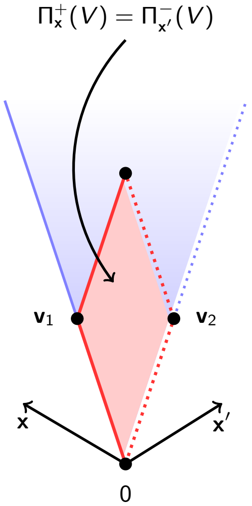
- ▷ obtain normal fundamental parallelepiped for $\mathbf{x} \in$ relint C



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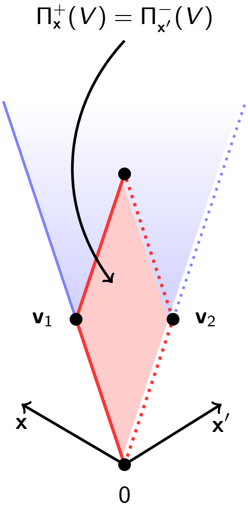

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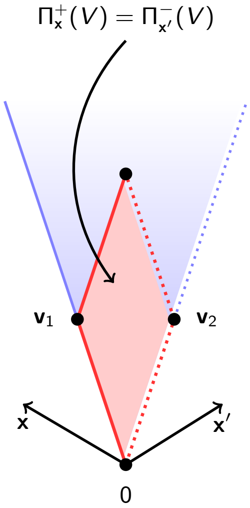

▶ Prop: $G_{C_{\mathbf{x}}^\pm}(\mathbf{t}) = \dfrac{G_{\Pi_{\mathbf{x}}^\pm}(\mathbf{t})}{(1-\mathbf{t}^{\mathbf{v}_0})(1-\mathbf{t}^{\mathbf{v}_1})\cdots(1-\mathbf{t}^{\mathbf{v}_d})}$

Ex

Half-Open Decompositions

- ▷ D general cone with triangulation $\mathcal{T}$,
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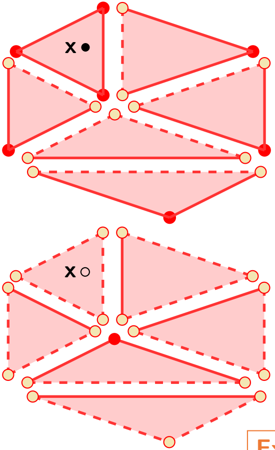

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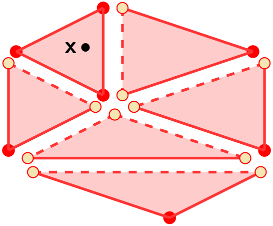

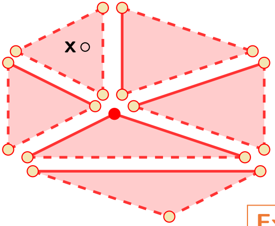

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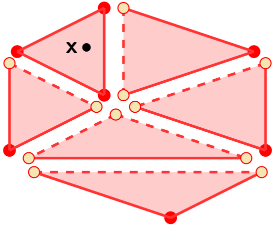

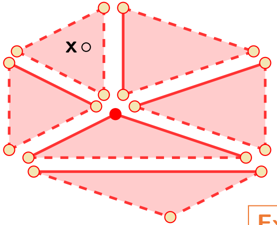

- ▷ D general cone with triangulation $\mathcal{T}$,
- ▷ $\mathbf{x} \in \mathbb{R}^d$ generic with respect to all cones in $\mathcal{T}$
- ▷ for $C \in \mathcal{T}$ let $V(C)$ be generators of C

- ▶ $\mathcal{T}[i]$: faces of dimension i in the triangulation

- ▶ Prop:
  - ▷ $D = \bigcup_{C \in \mathcal{T}[d]} C_{\mathbf{x}}^{+}$ and $\operatorname{relint} D = \bigcup_{C \in \mathcal{T}[d]} C_{\mathbf{x}}^{-}$ Ex
  - ▷ $\hat{G}_D(\mathbf{t}) = \sum_{C \in \mathcal{T}[d]} \hat{G}_{C_{\mathbf{x}}^{+}}(\mathbf{t})$ and $\hat{G}_{\operatorname{relint} D}(\mathbf{t}) = \sum_{C \in \mathcal{T}[d]} \hat{G}_{C_{\mathbf{x}}^{-}}(\mathbf{t})$ Ex
  - ▷ $G_D(\mathbf{t}) = \sum_{C \in \mathcal{T}[d]} \frac{G_{\Pi_{\mathbf{x}}^{+}(V(C))}(\mathbf{t})}{\prod_{v \in V(C)} (1 - \mathbf{t}^{\mathbf{v}})}$ and $G_{\operatorname{relint} D}(\mathbf{t}) = \sum_{C \in \mathcal{T}[d]} \frac{G_{\Pi_{\mathbf{x}}^{-}(V(C))}(\mathbf{t})}{\prod_{v \in V(C)} (1 - \mathbf{t}^{\mathbf{v}})}$ Ex

Half-Open Decompositions

▷ P lattice polytope witch cone $C(P)$

▷ $\mathcal{T}$ triangulation of P

$\longrightarrow$ triangulation extends to $C(P)$

▷ $\mathbf{x} \in \mathbb{R}^d$ generic with respect to all cones in $\mathcal{T}$

▶ Prop:

Ex

$$\mathrm{Ehr}_P(t) = \mathrm{G}_{C(P)}(t, \mathbf{1}) = \sum_{S \in \mathcal{T}[d+1]} \mathrm{G}_{S_\mathbf{x}^+}(t, \mathbf{1}) = \frac{\sum_{S \in \mathcal{T}[d+1]} \mathrm{G}_{\Pi(S)_\mathbf{x}^+}(t, \mathbf{1})}{(1-t)^{d+1}}$$

Non-Negativity

- Stanley's Non-negativity Theorem

  P lattice polytope, $\quad \mathrm{Ehr}_P(t) = \dfrac{h_0^\star + h_1^\star t + \cdots + h_d^\star t^d}{(1-t)^{d+1}}$

  Then $h_0^\star, \ldots, h_d^\star \; \geq \; 0$ and $h_0^\star \; = \; 1$

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- There is no similar result for the Ehrhart polynomial

  ▷ e.g. $S := \mathrm{conv}(0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + 18\mathbf{e}_3)$ has $\mathrm{ehr}_S(t) = 1 - t + t^2 + 3t^3$

Ehrhart-Macdonald-Reciprocity

- ▷ want to look at $\#(\operatorname{int} P \cap \mathbb{Z}^d)$

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- Theorem (Ehrhart-Macdonald Reciprocity)

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$$= (-1)^m t^{-(m+1)} \frac{1}{(1-\frac{1}{t})^{m+1}} = -\frac{1}{(1-t)^{m+1}}$$

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  C d-dimensional polyhedral cone with rational generators. Then

  $$\mathsf{G}_C(\mathbf{t}) \;=\; (-1)^d \mathsf{G}_{\mathrm{int}\, C}(\tfrac{1}{\mathbf{t}}) \qquad \left(\tfrac{1}{\mathbf{t}} \;=\; (\tfrac{1}{t_1}, \ldots, \tfrac{1}{t_d})\right)$$

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- ▷ plug this into

  $$G_C(\mathbf{t}) \;=\; \textstyle\sum_{S \in \mathcal{T}[d]} \mathrm{G}_{S^+_{\mathbf{x}}}(\mathbf{t}) \;=\; \sum_{S \in \mathcal{T}[d]} \frac{\mathrm{G}_{\Pi^+_{\mathbf{x}}(V)}(\mathbf{t})}{\prod_{\mathbf{v} \in V}(1 - \mathbf{t}^{\mathbf{v}})}$$

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proof:

$$\sum_{k\geq 1} \mathrm{ehr}_{\mathrm{int}\, P}(k) t^k = \hat{\mathrm{E}}\mathrm{hr}_{\mathrm{int}\, P}(t) = \mathrm{G}_{\mathrm{int}\, C(P)}(t, \mathbf{1}) = (-1)^{d+1} \mathrm{G}_{C(P)}(\frac{1}{t}, \mathbf{1})$$

$$= (-1)^{d+1} \hat{\mathrm{E}}\mathrm{hr}_P(\frac{1}{t}) = (-1)^{d+1} \sum_{k\geq 0} \mathrm{ehr}_P(k) \frac{1}{t^k}$$

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▷ compare coefficients

Brion's Theorem

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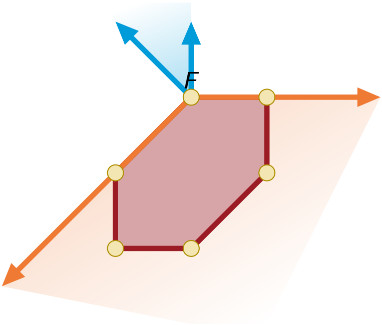
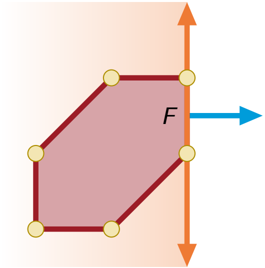
- P rational d-dimensional polytope
- F face of P, the tangential cone of F in P is

$$\mathrm{T}_F P := \{\mathbf{x} \mid \exists \mathbf{p} \in F, \varepsilon > 0 : \mathbf{p} + \varepsilon(\mathbf{x} - \mathbf{p}) \in P\}$$

$\longrightarrow$ up to a shift this cone is dual to the normal cone at this face





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coefficient on RHS: let $f_i := \#(i\text{-dimensional faces})$

$\mathbf{m}$ is in every tangent cone, so coefficient is

$$\textstyle\sum_{F \leq P} (-1)^{\dim F} \;=\; \sum_{i=0}^{d} (-1)^i f_i \;=\; 1$$

the last equality follows from Euler's identity

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coefficient on RHS:

▷ $S \;:=\; \{F \le P \mid \mathbf{m} \text{ beyond } F\}$

$$\longrightarrow \quad \mathbf{m} \in \mathsf{T}_F P \quad \Longleftrightarrow \quad F \notin S$$

▷ S is polyhedral complex with face vector $(g_0, \ldots, g_d)$

▷ coefficient is

$$\sum_{i=0}^{d} (-1)^i (f_i - g_i) \;=\; 1 - \sum_{i=0}^{d} (-1)^i g_i \;=\; 1 - 1 \;=\; 0$$

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▷ $P \;=\; C^+ \;+\; C^-$

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$$\mathrm{G}_P(t) = \Phi(\hat{\mathrm{G}}_{C^+}(t)) + \Phi(\hat{\mathrm{G}}_{C^-}(t)) = \Phi(\hat{\mathrm{G}}_{C^+}(t) + \hat{\mathrm{G}}_{C^-}(t))$$
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$$\begin{aligned} \mathrm{G}_P(t) \;&=\; \Phi(\hat{\mathrm{G}}_{C^+}(t)) \;+\; \Phi(\hat{\mathrm{G}}_{C^-}(t)) \;=\; \Phi(\hat{\mathrm{G}}_{C^+}(t) \;+\; \hat{\mathrm{G}}_{C^-}(t)) \\ &=\; \Phi(\hat{\mathrm{G}}_{\mathbb{R}}(t) \;+\; \hat{\mathrm{G}}_{P}(t)) \;=\; \Phi(\hat{\mathrm{G}}_{\mathbb{R}}(t)) \;+\; \mathrm{G}_P(t)\,. \end{aligned}$$

- ▷ this can only be true if $\Phi(\hat{\mathrm{G}}_{\mathbb{R}}(t)) \;=\; 0\,!$

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- Prop: $C \subseteq \mathbb{R}^d$ polyhedral cone
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- Theorem (Brion):

  $P \subseteq \mathbb{R}^d$ rational d-polytope

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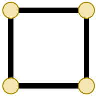
proof:
- ▷ Apply Φ to both sides of the Brianchon-Gram identity
- ▷ only vertex cones are pointed

Brion's Theorem

Example

- $P = \text{conv}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$



$$G_P(x,y) = \frac{1}{(1-x)(1-y)} + \frac{x}{(1-\frac{1}{x})(1-y)} + \frac{y}{(1-x)(1-\frac{1}{y})} + \frac{xy}{(1-\frac{1}{x})(1-\frac{1}{y})}$$

$$= \frac{1+(-x^2)+(-y^2)+x^2y^2}{(1-x)(1-y)}$$

$$= \frac{(1-x^2)(1-y^2)}{(1-x)(1-y)}$$

$$= 1+x+y+xy$$

Geometry of Numbers

Minkowski's Theorem

- Λ lattice of rank d
- Theorem (Blichfeldt)

  $S \subseteq \mathbb{R}^d$ (Lebesgue measurable) set with $\operatorname{vol} S > \det \Lambda$

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- ▷ Pick $\mathbf{a} \in S_{\mathbf{x}} \cap S_{\mathbf{y}}$ and let $\mathbf{p} := \mathbf{a} + \mathbf{x}$, $\mathbf{q} := \mathbf{a} + \mathbf{y}$.
- ▷ Then $\mathbf{p}, \mathbf{q} \in S$ and $\mathbf{p} - \mathbf{q}$ is a non-zero lattice point

Minkowski's Theorem

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  $K \subseteq \mathbb{R}^d$ centrally symmetric and convex, $\operatorname{vol} K > 2^d \det \Lambda$

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- for $k \geq 1$ let $\mathbf{x}_k \in \frac{k+1}{k} K \subset 2K$ be non-zero lattice point

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  is a non-zero lattice point
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  $K \subseteq \mathbb{R}^d$ centrally symmetric and convex, $\text{vol}\, K > 2^d \det \Lambda$

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  $\frac{1}{d!} 2^d \det \Lambda \leq \lambda_1 \cdot \lambda_2 \cdots \lambda_d \cdot \operatorname{vol} K \leq 2^d \det \Lambda$

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  - ▶ Prop: In fixed dimension we can find $\mathbf{w}$ in time
    polynomial time in $\log^3 \max(\|\mathbf{b}_i\|)$

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$$\varepsilon_i := \begin{cases} 0 & \text{if } \dim C_j < d \\ 1 & \text{if } \det(\mathbf{v}_1, \ldots, \mathbf{v}_d) = \det(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{w}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_d) \\ -1 & \text{otherwise.} \end{cases}$$

$$\longrightarrow \qquad \hat{G}_C(\mathbf{t}) \ = \ \textstyle\sum_{j=1}^d \varepsilon_j \, \hat{G}_{C_j}(\mathbf{t}) + \ \text{lower dimensional contributions}$$

- this decomposition has
  - ▷ at most d cones in dimension d
  - ▷ at most $2^d d$ cones in arbitrary dimension

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  $\longrightarrow$ n steps such that $\left(\text{index}(C)^{\frac{d-1}{d}}\right)^n < 2$

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- Theorem: $d \in \mathbb{Z}_>$ fixed. P d-dimensional polyhedron P given in exterior description.

  There is a polynomial time algorithm that computes $G_P(\mathbf{t})$ in the form

$$G_P(\mathbf{t}) = \sum_{i \in I} \varepsilon_i \frac{\mathbf{t}^{\mathbf{a}_i}}{(1 - \mathbf{t}^{\mathbf{v}_{i1}}) \cdots (1 - \mathbf{t}^{\mathbf{v}_{id}})}$$

  where ▷ $\varepsilon_i \in \{-1, 1\}$

  ▷ $\mathbf{a} \in \mathbb{Z}^d$

  ▷ $\mathbf{v}_{ij} \in \mathbb{Z}^d - \{\mathbf{0}\}$ for all i, j,

Short Basis

- Theorem(Lenstra, Lenstra, Lovász)

  Λ lattice, basis $\mathbf{b}_1, \ldots, \mathbf{b}_d$, $\ell := \max(\|\mathbf{b}_i\|)$.

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  $$\text{Then } \mathbf{w} = \sum_{i=1}^{d} \lambda_i \mathbf{v}_i \qquad \text{with} \qquad |\lambda_i| \leq \sqrt{d} M$$

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- This reduces finding $\mathbf{w}$ to a finite enumeration problem!

Theorem of Scott

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P lattice polygon with $i \geq 1$ interior lattice points. Then either

1. $P = 3\Delta_2$, so $\mathrm{vol}\, P = \frac{9}{2}$ and $i = 1$, or
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- proof

  can assume
  - ▷ P is contained in rectangle $\operatorname{conv}\begin{pmatrix} 0 & p' & p' & 0 \\ 0 & 0 & p & p \end{pmatrix}$
  - ▷ p minimal with this property
  - ▷ $2 \leq p \leq p'$

  q_b, q_t intersection with bottom and top

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- proof

  can assume
  - P is contained in rectangle $\operatorname{conv}\begin{pmatrix} 0 & p' & p' & 0 \\ 0 & 0 & p & p \end{pmatrix}$
  - p minimal with this property
  - $2 \leq p \leq p'$

  q_b, q_t intersection with bottom and top

  $$\longrightarrow \quad b \leq q_b + q_t + 2p \quad \text{and} \quad a \geq \frac{p(q_b + q_t)}{2} \quad \text{and} \quad a \geq \frac{b}{2}$$

Scott's Theorem

- $$b \le q_b + q_t + 2p$$
  $$a \ge \frac{p(q_b + q_t)}{2}$$
  $$a \ge \frac{b}{2}$$
- case distinction
  1. $p = q_b + q_t = 3$
  2. $p = 2$ or $q_b + q_t \ge 4$
  3. $p = 3$ and $q_b + q_t \le 2$
  4. $p \ge 4$ and $q_b + q_t \le 2$

$h^\star$ of polygons

- Theorem

  $h_2^\star t^2 + h_1^\star t + 1$ for $h_1^\star, h_2^\star \in \mathbb{N}$ is $h^\star$-polynomial of a lattice polygon if and only if

  1. $h_2^\star = 0$ and $h_1^\star$ is arbitrary. Then P has no interior lattice points.
  2. $h_2^\star = 1$ and $h_1^\star = 7$. Then $P \cong 3\Delta_2$.
  3. $1 \le h_2^\star \le h_1^\star \le 3h_2^\star + 3$. Then P has interior lattice points.

- proof

  ▷ $h_2^\star = i$ and $h_1^\star = b + i - 3$ and $a = \frac{1}{2}(1 + h_1^\star + h_2^\star)$

  (Scott) $\longrightarrow$ if $i \ge 1$ and $P \ne 3\Delta_2$, then $h_1^\star \le 3h_2^\star + 3$

  $b \ge 3$ $\longrightarrow$ if $i \ge 1$, then $h_2^\star = i \le b + i - 3 = h_1^\star$

  this proves necessity

$h^\star$ of polygons

- Theorem

  $h_2^\star t^2 + h_1^\star t + 1$ for $h_1^\star, h_2^\star \in \mathbb{N}$ is $h^\star$-polynomial of a lattice polygon if and only if

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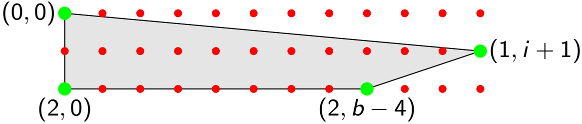
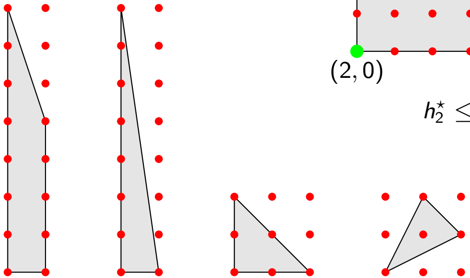
- proof



$$h_2^\star \leq h_1^\star \leq 3h_2^\star + 3 \iff 3 \leq b \leq 2i + 6$$



$h^\star$ of polygons

- Corollary

  $c_2 t^2 + c_1 t + 1$ with $c_1, c_2 \in \frac{1}{2}\mathbb{Z}$ and $c_1 \geq \frac{3}{2}$

  is Ehrhart polynomial of a lattice polygon P

  if and only if one of the following three conditions is satisfied:

  1. $c_1 - c_2 = 1$. Then P has no interior lattice points.
  2. $c_1 = c_2 = 9/2$. Then P is $3\Delta_2$.
  3. $c_1 \leq \frac{c_2}{2} + 2$. Then P has interior lattice points.