Defect Polytopes and Counter-Examples with polymake

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polymake

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- polytopes, point configurations
- simplicial complexes, tropical geometry, ...

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achievements in recent versions:

- lattice polytopes and relation toric geometry
- cones, fans, and polyhedral complexes
- callable library

polymake: Background

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▷ switch to demo

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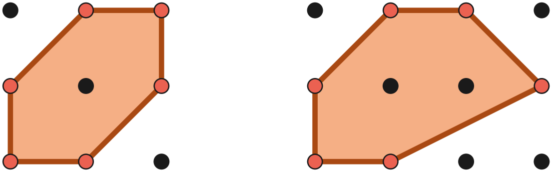
- interfaces to many other packages
  cdd, lrs, nauty, Singular, TOMPCOM, gfan
  normaliz, 4ti2, latte, ...

Lattice Polytopes

lattice polytope $P := \mathrm{conv}(V)$ for some $V \subset \mathbb{Z}^d$

$\longrightarrow$ associated toric variety X_P



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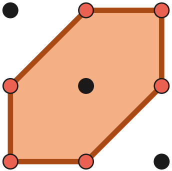

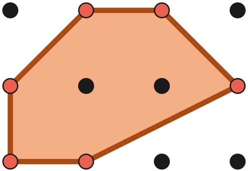

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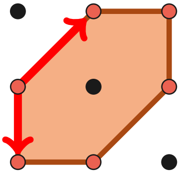

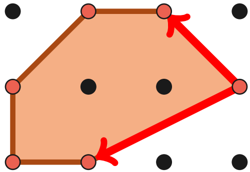

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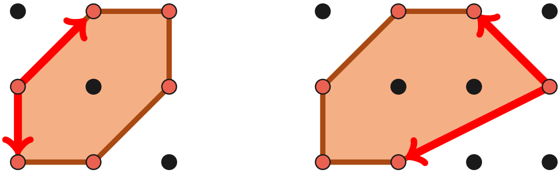
lattice volume of a face	LatticeVol(F)
	normalized volume w.r.t. $\Lambda = \text{span}_{\mathbb{Z}}(\text{aff}\, F \cap \mathbb{Z}^d)$



Defect Polytopes

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$$c_t(P) := \sum_{k=0}^{d} (-1)^{d-k} \frac{(k+t)!}{k!} \sum_{F \in \mathrm{F}_P(k)} \mathrm{LatticeVol}(F), \qquad t \geq 1$$

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Theorem: $c_t(P) \geq 0$ for all smooth lattice polytopes

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  my $v = $P->VERTICES; my $hd = $P->HASSE_DIAGRAM;
  my $sign = 1;  my $c = new Integer(0);
  for (my $d = $P->DIM; $d > 0; --$d) {
    my $lambda=$sign*fac($d+$t)/fac($d);
    foreach (@{$hd->nodes_of_dim($d)}) {
      my $F = new Polytope(VERTICES=>$v->minor($hd->FACES->[$_],All));
      $c += $lambda*$F->LATTICE_VOLUME;
    }
    $sign = -$sign;
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  $c += $sign*fac($t)*$P->N_VERTICES;
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Theorem: $C := [0,1]^3$, then $c_r(\mathrm{pyr}^r(C)) = -r! < 0$

polymake: where?

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- open source, GNU Public license
- latest release: version 2.10 (beta), released today
  - sources available at http://polymake.org
  - precompiled: rpm, deb, FreeBSD, (Mac app)
- online documentation at http://polymake.org
- user forum at http://forum.polymake.org
- this demo at http://polymake.org/doku.php/tutorial/a_determinants
- extensions: add and distribute your own objects, properties, rules, . . .