Defect Polytopes and Counter-Examples with polymake

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polymake: Software for research in discrete geometry
- polytopes, point configurations
- simplicial complexes, tropical geometry, ...
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- simplicial complexes, tropical geometry, ... 

- started in 1996: Michael Joswig and Ewgenij Gawrilow
- team: Katrin Herr, Sven Herrmann, Katja Kulas, Benjamin Lorenz, Silke Möser, Andreas Paffenholz, Thilo Rörig
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achievements in recent versions:

- lattice polytopes and relation toric geometry
- cones, fans, and polyhedral complexes
- callable library
polymake: Background

- interactive, perl based user interface
poly make: Background

- interactive, **perl** based user interface
- **objects**: Polytope<Rational>, Graph, FaceLattice, ...
polymake: Background

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- objects: Polytope<Rational>, Graph, FaceLattice, ...
- properties as class members (data or functions)
  - Polytope<Rational>: VERTICES, DIM, ...
  - Graph: EDGES, CONNECTIVITY, ...
  - FaceLattice: FACES, nodes_of_dim, ...
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▶ switch to demo
polyvake: Background

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- objects immutable:
  knowledge extended via **rules**
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- objects immutable:
  knowledge extended via rules

- interfaces to many other packages
cdd, lrs, nauty, Singular, TOMPCOM, gfan
normaliz, 4ti2, latte, ...
lattice polytope \( P := \text{conv}(V) \) for some \( V \subset \mathbb{Z}^d \) \( \longrightarrow \) associated toric variety \( X_P \)
lattice polytope

\[ P := \text{conv}(V) \] for some \( V \subset \mathbb{Z}^d \)

\( P \) smooth \( \rightarrow \) associated toric variety \( X_P \)

\( X_P \) smooth
lattice polytope  \( P := \text{conv}(V) \) for some \( V \subset \mathbb{Z}^d \)
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\( P \) smooth  \( X_P \) smooth

primitive edge directions at any vertex span \( \mathbb{Z}^d \)
a lattice polytope $P := \text{conv}(V)$ for some $V \subset \mathbb{Z}^d$ is associated with a toric variety $X_P$.

If $P$ is smooth, then $X_P$ is smooth.

Primitive edge directions at any vertex span $\mathbb{Z}^d$.

The lattice volume of a face $F$ is $\text{LatticeVol}(F)$.

The normalized volume with respect to $\Lambda = \text{span}_\mathbb{Z}(\text{aff } F \cap \mathbb{Z}^d)$ is defined.
\[ c_t(P) := \sum_{k=0}^{d} (-1)^{d-k} \frac{(k + t)!}{k!} \sum_{F \in F_P(k)} \text{LatticeVol}(F), \quad t \geq 1 \]
\[ c_t(P) := \sum_{k=0}^{d} (-1)^{d-k} \frac{(k + t)!}{k!} \sum_{F \in F_P(k)} \text{LatticeVol}(F), \quad t \geq 1 \]

For smooth \( P \): \( c_1 \) motivated from \textit{algebraic geometry}

\( X_P \) has dual defect \iff \( X_P^* \) is not a hypersurface

\( \leftrightarrow \quad c_1(P) = 0. \)
For smooth $P$: $c_1$ motivated from algebraic geometry

$X_P$ has dual defect $\iff$ $X_P^*$ is not a hypersurface

$\iff$ $c_1(P) = 0.$

$P$ is a defect polytope $:\iff$ smooth, $c_1(P) = 0.$
Defect Polytopes

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For smooth \( P \): \( c_1 \) motivated from algebraic geometry

\[ X_P \text{ has dual defect} \quad \iff \quad X_P^* \text{ is not a hypersurface} \]

\[ \iff \quad c_1(P) = 0. \]

\( P \) is a defect polytope \quad \iff \quad smooth, \( c_1(P) = 0. \)

Theorem: \( c_t(P) \geq 0 \) for all smooth lattice polytopes

\[ [t = 1 : \text{Gelfand, Kapranov, Zelevinsky}, t \geq 2: \text{di Rocco}] \]
Defect Polytopes

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\[ [t = 1 : \text{Gelfand, Kapranov, Zelevinsky}, \ t \geq 2 : \text{di Rocco}] \]

Conjecture: \( c_t(P) \geq 0 \) for all lattice polytopes \( P \)
Computing $c_t$

\[
c_t(P) := \sum_{k=0}^{d} (-1)^{d-k} \frac{(k + t)!}{k!} \sum_{F \in F_P(k)} \text{LatticeVol}(F)
\]

```perl
sub ct_invariant {
    my ($P, $t) = @_; my $v = $P->VERTICES; my $hd = $P->HASSE_DIAGRAM;
    my $sign = 1; my $c = new Integer(0);
    for (my $d = $P->DIM; $d > 0; --$d) {
        my $lambda=$sign*fac($d+$t)/fac($d);
        foreach (@{$hd->nodes_of_dim($d)}) {
            my $F = new Polytope(VERTICES=>$v->minor($hd->FACES->[$_],All));
            $c += $lambda*$F->LATTICE_VOLUME;
        }
        $sign = -$sign;
    }
    $c += $sign*fac($t)*$P->N_VERTICES;
    return $c;
}
```
Computing $c_t$

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```

▶ switch to demo
Defect Polytopes

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For smooth \( P \):

- \( c_1 \) motivated from algebraic geometry
- \( X_P \) has dual defect \( \iff \) \( X_P^* \) is not a hypersurface
- \( \iff c_1(P) = 0. \)

\( P \) is a defect polytope \( \iff \) smooth, \( c_1(P) = 0. \)

Theorem: \( c_t(P) \geq 0 \) for all smooth lattice polytopes

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Conjecture: \( c_t(P) \geq 0 \) for all lattice polytopes \( P \)
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**Theorem:** \( c_t(P) \geq 0 \) for all smooth lattice polytopes

\([t = 1 : \text{Gelfand, Kapranov, Zelevinsky}, t \geq 2 : \text{di Rocco}]\)

**Theorem:** \( C := [0, 1]^3 \), then \( c_r(\text{pyr}^r(C)) = -r! < 0 \)
polymake: where?

- open source, GNU Public license
- latest release: version 2.10 (beta), released today
  - sources available at http://polymake.org
  - precompiled: rpm, deb, FreeBSD, (Mac app)
- online documentation at http://polymake.org
- user forum at http://forum.polymake.org
- this demo at http://polymake.org/doku.php/tutorial/a_determinants
- extensions: add and distribute your own objects, properties, rules, ...