New Polytopes from Products

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Abstract. We construct a new 2-parameter family E_{mn}, $m, n \geq 3$, of self-dual 2-simple and 2-simplicial 4-polytopes, with flexible geometric realisations. E_{44} is the 24-cell. For large m, n the f-vectors have "fatness" close to 6.

The E_t-construction of Paffenholz and Ziegler applied to products of polygons yields cellular spheres with the combinatorial structure of E_{mn}. Here we prove polytopality of these spheres. More generally, we construct polytopal realisations for spheres obtained from the E_t-construction applied to products of polytopes in any dimension $d \geq 3$, if these polytopes satisfy some consistency conditions.

We show that the projective realisation space of E_{33} is at least nine dimensional and that of E_{44} at least four dimensional. This proves that the 24-cell is not projectively unique. All E_{mn} for relatively prime $m, n \geq 5$ have automorphisms of their face lattice not induced by an affine transformation of any geometric realisation. The group $\mathbb{Z}_m \times \mathbb{Z}_n$ generated by rotations in the two polygons is a subgroup of the automorphisms of the face lattice of E_{mn}. However, there are only five pairs (m, n) for which this subgroup is geometrically realisable.

Introduction

In 2003, Eppstein, Kuperberg, and Ziegler introduced a new method for the construction of 2-simple and 2-simplicial 4-polytopes [EKZ03]. This was subsequently extended to arbitrary dimensions and to spheres and lattices in [PZ04]. The construction produces PL $(d-1)$-spheres from d-polytopes by subdividing and combining faces of the polytope in a certain way. It is unknown whether these spheres are polytopal in general. However, [PZ04] lists several series of examples in which they have a polytopal realisation.

Here we provide sufficient conditions for the polytopality of the spheres that we obtain when the construction is applied to products of two polytopes. We present examples of d-dimensional products for which these conditions are satisfied, for all $d \geq 3$. Our main interest is in the application to products $C_m \times C_n$ of two polygons with m and n vertices. We prove that these products satisfy our conditions for all $m, n \geq 3$, resulting in a two-parameter family E_{mn} of self-dual, 2-simplicial and 2-simple polytopes. All these polytopes have a large combinatorial symmetry group and only three different combinatorial types of vertices and facets.

The underlying CW spheres in the special case $m = n$ were described earlier by Gévay [Gev04] and Bokowski [Bok04]. Gévay also considered symmetry properties of these spheres. Polytopality for $\frac{1}{m} + \frac{1}{n} \geq \frac{1}{2}$ is a consequence of a theorem of Santos [San00, Rem. 13].

There are two different notions of symmetry for a polytope: (1) automorphisms of the face lattice (combinatorial symmetries), and (2) transformations that setwise preserve a geometric realisation of the polytope (geometric symmetries). Any geometric symmetry preserves incidences and thus induces a combinatorial symmetry. However, the opposite implication is not true in general, i.e. not all combinatorial symmetries of a polytope can always be realised geometrically in some realisation of the polytope. Mani [Man71] and Perles [Grü03, p. 120] proved that all 3-polytopes, and all d-polytopes with at most $d + 3$ vertices, have a geometric realisation whose geometric symmetry group is isomorphic to the combinatorial one, while Bokowski, Ewald, and Kleinschmidt [BEK84] presented a 4-polytope on 10 vertices having a combinatorial symmetry not induced by a geometric one.

Here we prove that all polytopes E_{mn} for relatively prime $m, n \geq 5$ have geometrically non-realisable combinatorial symmetries. Furthermore, the combinatorial symmetry group of E_{mn} always contains the product $\mathbb{Z}_m \times \mathbb{Z}_n$ of two cyclic groups induced by a rotation of the vertices in the two polygons. However, there are only five pairs (m, n) in which the geometric symmetry group of some realisation has a subgroup inducing these combinatorial symmetries.

The polytope E_{44} is combinatorially equivalent to the 24-cell, and applying the E-construction to the product of two unit squares produces its regular realisation. However, our polytopality conditions for the E-construction of products allow for much more flexibility. For the smallest instance E_{33} we work out all degrees of freedom that our conditions permit and give an explicit construction of all possible such realisations. This will prove that the projective realisation space $\mathcal{R}_{proj}(E_{33})$ of E_{33} is at least nine dimensional. For the 24-cell we present a simple 4-parameter family of realisations showing that $\mathcal{R}_{proj}(E_{44})$ is at least four dimensional. In particular, the 24-cell is projectively not unique (cf. [McM76]).

Eppstein, Kuperberg, and Ziegler [EKZ03] introduced the "fatness" $F(P)$ of a 4-polytope P, which is roughly the quotient of the number of edges and ridges by the number of vertices and facets. They construct an example with fatness approximately 5.073. For large m, n our polytopes E_{mn} will have fatness arbitrarily close to 6. However, Ziegler [Zie04] recently constructed a new family of 4-polytopes from projections of products of polygons whose fatness approaches 9.

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1 POLYTOPES, PRODUCTS, AND THE E-CONSTRUCTION

This section gives a short introduction to polytopes, their products, and the E-construction. See [Zie95] and [PZ04] for more background.

Polytopes. A *polytope* P is the convex hull of a finite set of points in $\mathbb{R}^n$. Its *dimension d* is the smallest dimension of an affine subspace containing P. $V(P)$ denotes the *set of all vertices* of a d-polytope P. Faces of codimension 1 and 2 are called *facets* and *ridges*. Let f_S for $S \subset \{0, \ldots, d-1\}$ be the number of increasing

chains with one face of dimension i for each $i \in S$. The vector collecting these numbers is called the *flag vector* flag(P). The *f-vector* is the subset of flag(P) corresponding to the entries with $|S| = 1$. We set $f_d(P) := 1$.

Products. For $i = 0, 1$ let P_i be d_i-polytopes with flag vectors flag$(P_i) = (f_S(P_i))_{S \subseteq \{0,\ldots,d_i-1\}}$. The *product* $P_0 \times P_1$ is the convex hull of

$$V(P_0 \times P_1) := \{(v, w) \in \mathbb{R}^{d_0+d_1} \mid v \in V(P_0), w \in V(P_1)\}.$$

Equivalently, $P_0 \times P_1 := \{(v, w) \in \mathbb{R}^{d_0+d_1} \mid v \in P_0,\ w \in P_1\}$. It has dimension $(d_0 + d_1)$ and flag vector flag$(P_0 \times P_1) := (f_S(P_0 \times P_1))_{S \subseteq \{0,\ldots,d_0+d_1-1\}}$ with

$$f_S(P_0 \times P_1) := f_{(s_1,s_2,\ldots,s_k)}(P_0 \times P_1)$$
$$= \sum_{u_1+v_1=s_1} \sum_{u_2+v_2=s_2} \cdots \sum_{u_k+v_k=s_k} f_{(u_1,u_2,\ldots,u_k)}(P_0) f_{(v_1,v_2,\ldots,v_k)}(P_1).$$

In this formula we set $f_{(t_1,t_2,\ldots,t_k)} \equiv 0$ unless $t_1 \leq t_2 \leq \ldots \leq t_k$ and define $f_{(t_1,t_2,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_k)} := f_{(t_1,t_2,\ldots,t_{i-1},t_{i+1},\ldots,t_k)}$ if $t_i = t_{i+1}$.

We have defined here the *geometric (orthogonal)* product as the convex hull of all pairs of geometrically given vertices. A more general definition would just require a polytope combinatorially equivalent to this.

E-Construction. For our purposes the E-construction of a d-polytope P, $d \geq 2$, is best viewed as a construction that takes polytopes as input and produces regular CW spheres (their "E-sphere") from them. The original definition in [PZ04] depends on a parameter t between 0 and $d - 1$ (the dimension of "distinguished" elements). We omit this parameter in the notation, as we use only the case $t = d-2$.

Here is the construction. Let P be a d-polytope. The E-construction assigns to P a *CW*-sphere $E(P)$ by the following two steps:

(1) *Stellarly subdivide all facets of the polytope P,*

(2) *and merge facets of the subdivision sharing a ridge of P.*

Each facet of the subdivision contains precisely one such ridge, so we merge pairs of facets of the subdivision. Thus, combinatorially the facets of $E(P)$ are bipyramids over the ridges of P. See Figure 1.1 and Figure 1.2 for a two-dimensional and a three-dimensional example of this construction. In dimensions $d \geq 3$ all vertices of the original polytope are preserved, while for $d = 2$ they lie on the new edges. A more formal definition given on the level of face lattices is in [PZ04, Def. 1.2]. For any polytope P the sphere $E(P)$ is a piecewise linear CW sphere [PZ04, Thm. 2.1]. If these spheres are polytopal then we call the resulting polytope the E-*polytope* of P. This is e.g. the case for all dual-to-stacked 4-polytopes [PZ04, Sect. 3]. The f-vector of $E(P)$ is given by

$$f_k(E(P)) := \begin{cases} f_{d-2}(P) & k = d - 1 \\ f_{d-3,d-1}(P) & k = d - 2 \\ f_k(P) + f_{k-1,d-1}(P) & \text{otherwise,} \end{cases}$$

where we set $f_{-1j} := f_j$.

In the above definition the E-polytope of some polytope P just denotes some polytope being combinatorially equivalent to the sphere obtained from P via the E-construction. In the following we need a stricter version of the connection between P and its E-polytope.

Definition 1.1 (vertex-preserving). *A polytopal realisation of $E(P)$ for a given geometrically realised polytope P is called* vertex-preserving *if it is obtained from the realisation of P by placing new vertices beyond the facets of P and taking the convex hull. See Figure 1.3 for an example.*

Remark 1.2. For illustrations we will sometimes also apply the E-construction to a 1-polytope S (i.e. a segment). In this case $E(S)$ is defined to be a segment containing S in its interior.

Polygons. We denote a (convex) polygon with m vertices $v_0, \ldots, v_{m-1}$ by C_m. We usually assume that the vertices are numbered consecutively and take indices modulo m.

By E_{mn} we denote the result of the E-construction applied to the product $C_m \times C_n$ of an m-gon and an n-gon. This is a 4-dimensional 2-simplicial and 2-simple CW sphere. The flag vectors of $C_m \times C_n$ and E_{mn} are:

$$\text{flag}(C_m \times C_n) = (mn, 2mn, mn + m + n, m + n; 4mn)$$
$$\text{flag}(E_{mn}) = (mn + m + n, 6mn, 6mn, mn + m + n; 8mn + 2(m+n)), \qquad (1)$$

where we have only recorded the values $(f_0, \ldots, f_3; f_{03})$. All other entries of the flag vector follow from the generalised Dehn-Sommerville equations [BB85].

2 The E-construction of products

Let P_0, P_1 be two polytopes of dimensions d_0 and d_1. We give sufficient conditions for the existence of a polytopal realisation of the sphere $E(P_0 \times P_1)$ obtained from the polytope $P_0 \times P_1$. If we restrict to vertex-preserving realisations, then these conditions are also necessary. The conditions are the following:
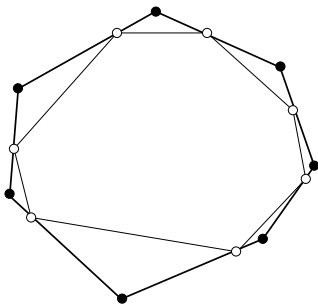

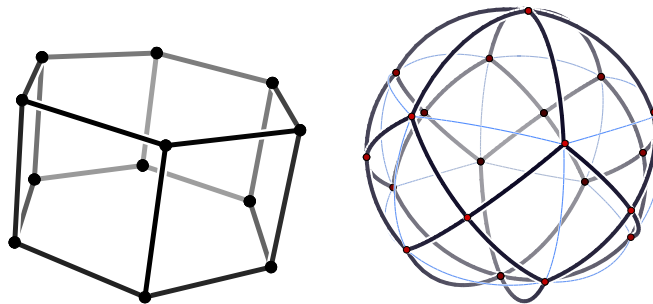*Fig. 1.1:* The E-construction (thick edges) applied to a polygon (thin edges).

Fig. 1.2: A polytope and its E-sphere (in bold, the polytope is drawn thin to show the old ridges).

(A) There exist vertex-preserving realisations of $E(P_0)$ and $E(P_1)$.

(B) For $i = 0, 1$ let $T_i := V(E(P_i)) \setminus V(P_i)$. There are two maps

$$\beta_i : T_i \to \text{int}(P_{1-i})$$

such that for any $(v_0, v_1) \in T_0 \times T_1$ the fraction of the segment $|v_0, \beta_1(v_1)|$ outside P_0 equals the fraction of the segment $|v_1, \beta_0(v_0)|$ inside P_1.

Theorem 2.1. *Let P_0, P_1 be a pair of polytopes with $\dim(P_0 \times P_1) \geq 3$ that satisfies (A) and (B). Let $S \subset \mathbb{R}^{d_0} \times \mathbb{R}^{d_1}$ be the point set containing the following points:*

(a) all pairs (p_0, p_1) for $p_0 \in V(P_0)$, $p_1 \in V(P_1)$,

(b) all pairs $(v_0, \beta_0(v_0))$ for $v_0 \in T_0$,

(c) all pairs $(\beta_1(v_1), v_1)$ for $v_1 \in T_1$.

Then $\text{conv}(S)$ is a vertex-preserving polytopal realisation of $E(P_0 \times P_1)$. Moreover, for the existence of vertex-preserving realisations of $E(P_0 \times P_1)$ the two conditions (A) and (B) are both necessary and sufficient.

See Figure 2.1 for an example of two triangles satisfying (A) and (B).

Proof of Theorem 2.1. The proof has two parts. First we prove the necessity of the two conditions (A) and (B) for vertex preserving realisations and then their sufficiency.

Let P_0 and P_1 be two geometrically realised polytopes of dimension d_0 and d_1 with $d_0 + d_1 \geq 3$. Suppose $E(P_0 \times P_1)$ exists and is a vertex-preserving realisation of $P_0 \times P_1$. We can split the vertex set of $E(P_0 \times P_1)$ into the vertex set of $P_0 \times P_1$ and a set consisting of one vertex beyond each facet of $P_0 \times P_1$.

Define standard projections $\pi_j : \mathbb{R}^{d_0+d_1} \longrightarrow \mathbb{R}^{d_j}$ for $j = 0, 1$. By assumption, the vertex set of P_j is contained in $\pi_j(V(E(P_0 \times P_1)))$ for $j = 0, 1$. We determine the images of the other vertices of $E(P_0 \times P_1)$ under π_0 and π_1.

The facets of the product $P_0 \times P_1$ are of the form (1) "Facet of P_0" $\times P_1$ or (2) $P_0 \times$ "Facet of P_1". Thus, we have two different types of ridges: (I) Those between two
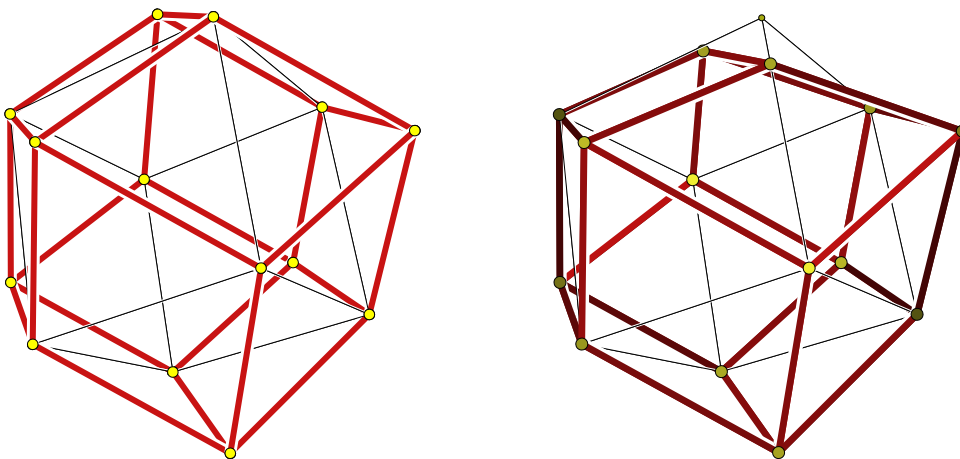
Fig. 1.3: The left realisation of $E(C)$ of the unit cube C is vertex-preserving, the right is not: observe the top vertex of the cube (and there is no cube for which it is).

adjacent facets of the first or second type, and (II) those between a facet of the first and one of the second type. We deal with these two cases separately:

(I) Let F and F' be two adjacent facets of the first type and v, v' the two vertices of $E(P_0 \times P_1)$ beyond F and F'. Let R be the ridge between F and F'. The projections $\pi_0(F)$ and $\pi_0(F')$ are adjacent facets of P_0 with common ridge $\pi_0(R)$. $\pi_0(v)$ and $\pi_0(v')$ are points beyond these facets. v, v' and R lie on a common (facet defining) hyperplane H of $E(P_0 \times P_1)$ in $\mathbb{R}^{d_0+d_1}$. So the points $\pi_0(v)$, $\pi_0(v')$ and the ridge $\pi_0(R)$ all lie on the hyperplane $\pi_0(H)$ in $\mathbb{R}^{d_0}$ and $\pi_0(H)$ defines a face of $\pi_0(E(P_0 \times P_1))$, which must in fact be a facet. Thus, the convex hull of the projection of all vertices of $E(P_0 \times P_1)$ is $E(P_0)$. Similarly, projecting with π_1 gives $E(P_1)$.

(II) Let w_0 and w_1 be two vertices of $E(P_0 \times P_1)$, the first beyond a facet $G_0 \times P_1$, the second beyond $P_0 \times G_1$, where G_0 and G_1 are facets of P_0 and P_1. Let $R = G_0 \times G_1$ be the ridge between these two facets. The segment s between w_0 and w_1 intersects R in a point q. $\pi_0(q)$ is contained in G_0 and $\pi_1(q)$ is contained in G_1. So $\pi_0(w_1)$ is contained in the interior of P_0 and $\pi_1(w_0)$ in the interior of P_1. Projections preserve ratios, so

$$r := \frac{|w_0 q|}{|w_0 w_1|} = \frac{|\pi_0(w_0)\pi_0(q)|}{|\pi_0(w_0)\pi_0(w_1)|} = \frac{|\pi_1(w_0)\pi_1(q)|}{|\pi_1(w_0)\pi_1(w_1)|}.$$

Hence, a vertex-preserving realisation of $E(P_0 \times P_1)$ implies the conditions (A) and (B). This proves the necessity part of the theorem.

Now we prove sufficiency of (A) and (B). Suppose we have, according to (A) and (B), constructed $E(P_0)$ and $E(P_1)$ together with the maps $\beta_i : T_i \to \text{int}(P_{1-i})$, $T_i := V(E(P_i)) \setminus V(P_i)$, $i = 0, 1$, and have formed the set S defined in the theorem. We have to show that all facets of the convex hull of S defined thereby are bipyramids over ridges of $P \times P_1$ and that there is precisely one vertex of S beyond each facet of $P_0 \times P_1$. There are two different cases to consider:
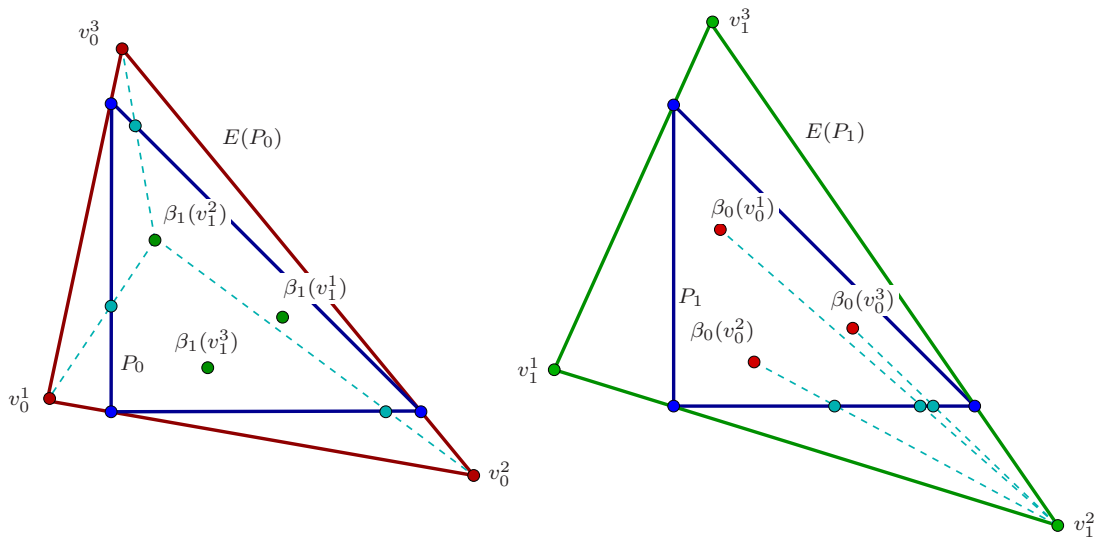

Fig. 2.1: Realising the product of two triangles

(I) Let R be a ridge of P_0. Then $R \times P_1$ is a ridge of $P_0 \times P_1$. Let F and F' be the two facets of P_0 adjacent to R and p, p' the vertices of $E(P_0)$ above F and F' (see Figure 2.2). Let v be the facet normal of the facet F_E of $E(P_0)$ formed by R, p and p' and let $l := \langle v, p \rangle$.

By construction, the points $(p, \beta_0(p))$, $(p', \beta_0(p'))$ and (r, q) for $r \in V(R)$ and $q \in V(P_1)$ are contained in the hyperplane $H := \{x \mid \langle (v, \mathbf{0}), x \rangle = l\}$, where $\mathbf{0}$ is the d_1-dimensional zero vector. All points of $V(E(P_0)) \setminus (V(R) \cup \{p, p'\})$ are on the same side of the hyperplane defined by the facet F_E. So all points in

$$V(E(P_0 \times P_1)) \setminus (V(R \times P_1) \cup \{(p, \beta_0(p)), (p', \beta_0(p'))\})$$

are on the same side of the hyperplane H and

$$\mathrm{conv}(V(R \times P_1) \cup \{(p, \beta_0(p)), (p', \beta_0(p'))\})$$

is a facet of $E(P_0 \times P_1)$. The same argument applies to ridges of type $P_0 \times R$ for ridges R of P_1.

(II) Now consider a ridge of type $F_0 \times F_1$ for a facet F_0 of P_0 and a facet F_1 of P_1. Let p_0 be the vertex of $E(P_0)$ beyond F_0 and p_1 the vertex of $E(P_1)$ beyond F_1. Let i_0 be the intersection point of the segment between p_0 and $\beta_1(p_1)$ and the facet F_0, and i_1 the intersection point of the segment between p_1 and $\beta(p_0)$ and the facet F_1. By construction we have

$$\frac{|p_0, i_0|}{|p_0, \beta_1(p_1)|} = \frac{|\beta_0(p_0), i_1|}{|\beta_0(p_0), p_1|}$$

and the point $(p_0, \beta_0(p_0))$ is contained in the line defined by $(\beta_1(p_1), p_1)$ and (i_0, i_1). So the points $V(F_0 \times F_1)$, $(p_0, \beta_0(p_0))$ and $(\beta_1(p_1), p_1)$ lie on a common hyperplane H.

The product $P_0 \times P_1$ lies entirely on one side of H by construction. Suppose there is a point x of S on the other side of H. As H is a valid hyperplane for the ridge $F_0 \times F_1$, any point beyond it is also beyond either the facet hyperplane of $F_0 \times P_1$ or $P_0 \times F_1$. Assume the first. For any $z \in S$ we have either $\pi_0(z) \in S_1$, or $\pi_0(z) \in V(P_0)$, or $\pi_0(z) \in V(E(P_0)) \setminus V(P_0)$. $x \in S$ is beyond $F_0 \times P_1$, therefore only $\pi_0(x) \in V(E(P_0)) \setminus V(P_0)$ is possible. Thus, $\pi_0(x)$ is the unique vertex of $E(P_0)$ beyond F_0, so $\pi_0(x) = p_0$ and $x \in H$.

This proves that the two conditions (A) and (B) are sufficient for the existence of a polytopal realisation of $E(P_0 \times P_1)$. $\square$
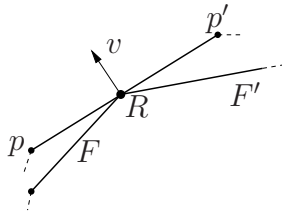

Fig. 2.2: Sufficiency: The case of "ridge $\times$ polytope"

In Section 4.4 we present some general applications of Theorem 2.1. However, we mostly use a more restrictive version of it. We tighten the conditions (A) and (B) in the following way to make them more manageable:

(A′) There exist vertex-preserving realisations of $E(P_0)$ and $E(P_1)$.

(B′) For $i = 0, 1$ let $T_i := V(E(P_i)) \setminus V(P_i)$. There are points $s_i \in \text{int}(P_i)$ and some $0 < r < 1$ such that for any $v_0 \in T_0$ and $v_1 \in T_1$

$$r|s_0, v_0| = |s_0, q_0| \qquad (1 - r)|s_1, v_1| = |s_1, q_1|,$$

where q_i is the intersection of the segment from s_i to v_i and $|a, b|$ denotes the length of the segment from a to b (Hence $\beta_i(x) \equiv s_i$ for $i = 0, 1$).

Theorem 2.1 immediately implies the following.

Corollary 2.2. *Let P_0, P_1 be a pair of polytopes with $\dim(P_0 \times P_1) \geq 3$ that satisfies (A) and (B). Let $S \subset \mathbb{R}^{d_0} \times \mathbb{R}^{d_1}$ be the point set containing the following points:*

(a) *all pairs (p_0, p_1) for $p_0 \in V(P_0)$, $p_1 \in V(P_1)$,*

(b) *all pairs (v_0, s_1) for $v_0 \in T_0$,*

(c) *all pairs (s_0, v_1) for $v_1 \in T_1$.*

Then $\text{conv}(S)$ is a vertex-preserving polytopal realisation of $E(P_0 \times P_1)$. $\qquad\square$

In this setting, the only connection between the construction of the two factors is the value r of the ratio. Thus, if we construct a polytope P together with a vertex-preserving realisation of $E(P)$ and a single point s in its interior such that all segments from s to the vertices of $E(P)$ not in P intersect ∂P with ratio r, then we can combine this with any other such instance for a ratio of $1 - r$.

3   Explicit realisations

Now we apply the construction of the previous section and produce examples of products of polytopes with a realisation of their E-polytope. The main focus is on the realisation of the E-polytope E_{mn} of a product of an m-gon and an n-gon. We produce polytopal realisations for all of them. Afterwards we briefly discuss examples in dimensions $d \geq 5$. For some of the examples explicit polymake (Gawrilow and Joswig [GJ00]) data are available on request.

3.1   Products of Polygons

We begin with products of polygons and present a method to obtain a "flexible" geometric realisation of $E_{mn} := E(C_m \times C_n)$ for all $m, n \geq 3$. We will discuss degrees of freedom in this construction in Section 4.4.

Theorem 3.1. *The CW spheres E_{mn} are polytopal for all $m, n \geq 3$.*

In the five cases when m, n satisfy $\frac{1}{m} + \frac{1}{n} \geq \frac{1}{2}$, polytopality follows from a construction of Santos [San03], [San00, Rem. 13]. These realisations are presented in Theorem 4.3.

We use the restricted setting of Corollary 2.2 for the proof and construct only one of the two factors, but subject to the conditions (A′) and (B′). We make the following definition for this.

Definition 3.2. *Let $\frac{1}{3} < r < \frac{2}{3}$. By $D(k,r)$ we denote a realisation of a k-gon C_k together with*

> ▷ *a distinguished inner point s,*
> ▷ *a vertex-preserving E-polytope $E(C_k)$, such that segments from s to vertices of $E(C_k)$ are intersected by the boundary of C_k with ratio r.*

See Figure 3.1 for an example. To realise E_{mn} we choose a ratio r between $\frac{1}{3}$ and $\frac{2}{3}$ and combine the points of $D(m,r)$ and $D(n, 1-r)$ according to Corollary 2.2. We restrict to $m \geq 4$ in the following and refer to the proof of Theorem 4.3 for the case $m = 3$.

The next construction is illustrated in Figure 3.2. Let Γ denote the graph of the parabola $x \mapsto x^2$ in the plane $\mathbb{R}^2$ with coordinates x and y. We construct the polygon C_m such that all but one of the vertices lie on Γ, and $E(C_m)$ such that all but one of the edges are tangent to Γ.

Let s be the point $(0,1)$ and define the three functions

$$\mathcal{C}(x) := \frac{x + \sqrt{(1-r)(2r + rx^2 + x^2)}}{r}$$

$$\overline{\mathcal{C}}(x) := \frac{x(x + \sqrt{x^2 + 2r^2x^2 - 2rx^2 - 2r + 2r^2 - rx})}{r}$$

$$\mathcal{E}(x) := \frac{x + \sqrt{x^2 + 2r^2x^2 - 2rx^2 - 2r + 2r^2}}{2r}.$$

For any $a \geq 0$ let $p(a) \in \mathbb{R}^2$ be the intersection point of the tangents to Γ in (a, a^2) and $(\mathcal{C}(a), \mathcal{C}(a)^2)$. We have the following facts about these functions.

Lemma 3.3. *Let $0 < r < \frac{2}{3}$.*

1. *For any $a \geq 0$ the secant line between (a, a^2) and $(\mathcal{C}(a), \mathcal{C}(a)^2)$ intersects the segment between s and $p(a)$ in a point $q(a)$ satisfying*

$$r|s, p(a)| = |s, q(a)|,$$

where $|x_0, x_1|$ denotes the length of the segment between x_0 and x_1.

2. *For any $a \geq 1$ the line between (a, a^2) and $(0, \overline{\mathcal{C}}(a))$ intersects the segment between s and $\overline{p}(a) := (\overline{\mathcal{C}}(a), \mathcal{E}(a))$ in a point $\overline{q}(a)$ satisfying*

$$r|s, \overline{p}(a)| = |s, \overline{q}(a)|.$$

3. *For any $a \geq 0$ we have $\mathcal{C}(a) > 1$.*
4. *For any $a > 1$ we have $\overline{\mathcal{C}}(a) > a^2$.*
5. *For any $a > 1$ we have $\overline{\mathcal{C}}(x) = 2x\mathcal{E}(x) - x^2$.* $\qquad\square$

With this information at hand we are ready to give an iterative construction for $D(m,r)$ in the case $m \geq 4$ and $0 < r < \frac{2}{3}$. We distinguish the two cases m even and m odd.

▷ For m even we choose the point $v_0^+ := (0, 0)$ as our first vertex of C_m, for m odd we take the two points $v_0^\pm := (\pm\sqrt{\frac{1-r}{1+r}}, \frac{1-r}{1+r})$.

▷ In the i-th step we extend with the points $v_i^\pm := (\pm\mathcal{C}(a), \mathcal{C}(a)^2)$, where a is the x-coordinate of v_{i-1}^+.

▷ We repeat the previous step until we have constructed $m - 1$ points of C_m.

▷ In the last step we add the vertex $v_{\lfloor\frac{m}{2}\rfloor} := (0, \overline{\mathcal{C}}(a))$, where a is the x-coordinate of $v_{\lfloor\frac{m}{2}\rfloor - 1}$.

The edges of $E(C_m)$ are the tangents to Γ in the vertices of C_m, except for $v_{\lfloor\frac{m}{2}\rfloor}$, where we take the horizontal line running through $v_{\lfloor\frac{m}{2}\rfloor}$. This line intersects the tangents to $(v_{m-1}^\pm, (v_{m-1}^\pm)^2)$ in the points $(\pm\mathcal{E}(v_{m-1}^+), \overline{\mathcal{C}}(v_{m-1}^+))$.

Lemma 3.3 guarantees the condition on the intersection ratio. The point $s = (0, 1)$ is inside C_m by Lemma 3.3(3),(5). Hence we can construct $D(m, r)$ for $0 < r < \frac{2}{3}$ and $m \geq 4$. Finally, a realisation for $m = 3$ and $\frac{1}{4} < r < 1$ is given in the proof of Theorem 4.3. By combining $D(n, r)$ with $D(m, 1 - r)$ for some $m \geq 3$ we obtain E_{mn}. This proves Theorem 3.1.

Remark 3.4. $D(m, r)$ can in fact be constructed for any r between 0 and 1 and any $m \geq 3$, but the formulas for the vertices and the cases one has to distinguish in the construction tend to get complicated rather quickly.
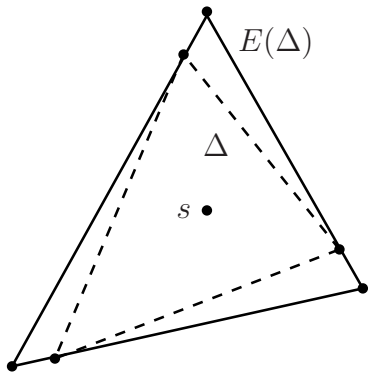

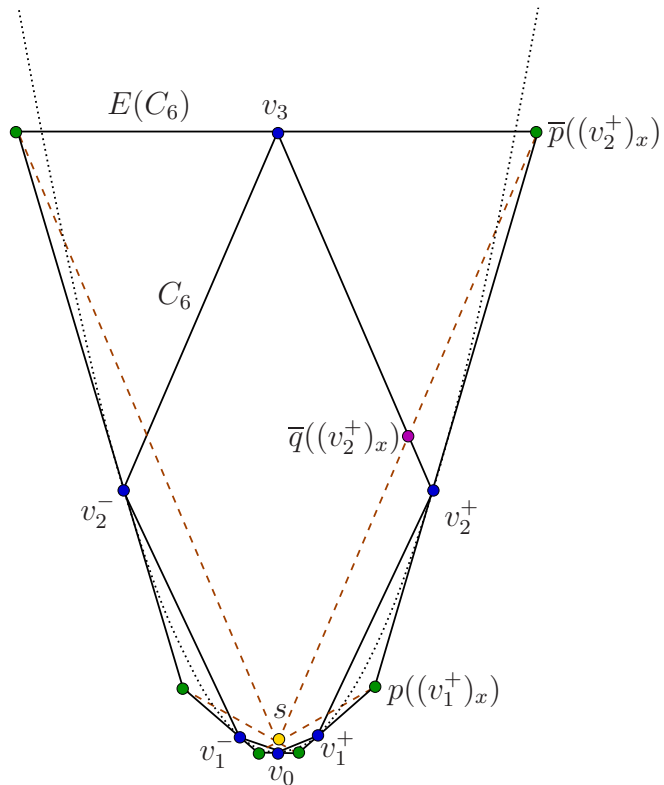Fig. 3.1: An example of a realisation of $D(3, \frac{1}{3})$

Fig. 3.2: The construction of $D(6, \frac{3}{5})$. Note that the x-axis has larger scale than the y-axis

3.2 Some higher-dimensional examples

Satisfying the conditions (A) and (B) is more difficult if the two factors P_0 and P_1 have more facets. Thus in higher dimensions and for "more complex" polytopes, it is usually hard to find maps β_0 resp. β_1, unless one can exploit some kind of symmetry.

There are, however, two obvious families of polytopes that we can choose as factors of a product polytope: The d-simplex Δ_d and the d-cube C_d. Both can be realised together with their E-construction satisfying even the restrictive conditions of Corollary 2.2.

▷ The cube with its E-construction and an intersection ratio of $r = \frac{1}{2}$ can be realised as follows: For C_d we take the standard ± 1-cube. The new vertices for the E-polytope are $\pm 2 \cdot e_i$, where e_i are the standard unit basis vectors. The origin is an inner point s satisfying all requirements.

▷ The construction for the d-simplex Δ_d is slightly more difficult. We give an inductive construction that produces realisations for any ratio $\frac{1}{2} \leq r < 1$, that is, at least half of the segment is inside Δ (where r is the parameter appearing in the conditions in the box on page 5). We can clearly construct such a realisation for a triangle, i.e. for a simplex of dimension $d = 2$.

For $d > 2$ we take a regular realisation Δ of the simplex and a scaled version $\Delta' := \frac{1}{r} \cdot \Delta$ with the same barycentre. We choose one facet F of Δ and the corresponding scaled facet F' in Δ'. Place the first new vertex v in the barycentre of F'. The vertices of any ridge R of F together with the point v uniquely define a hyperplane. F has d ridges, so we obtain d different hyperplanes $H_1, \ldots, H_d$ by this. $H_1, \ldots, H_d$ intersect all facet hyperplanes of Δ', except that to F', in codimension-2-planes that lie in a common hyperplane H. H is parallel to F.

H cuts Δ and Δ' in two simplices $\tilde{\Delta}$ and $\tilde{\Delta}'$ of dimension $d - 1$. (Recall that $r \geq \frac{1}{2}$, so H intersects Δ below the barycentre if viewed from F.) $\tilde{\Delta}'$ is (viewed in the hyperplane H) a scaled version of $\tilde{\Delta}$ with a scaling factor $\frac{1}{r'} \leq \frac{1}{r}$. By induction, we have a solution for the corresponding problem for $\tilde{\Delta}$ and $r' \geq r \geq \frac{1}{2}$ in H (where the inner point is the projection of the barycentre of Δ).

These points, together with the one vertex v chosen before, give a realisation of $E(\Delta)$ that satisfies the conditions of Corollary 2.2. See Figure 3.3 for the case $d = 3$.

We can combine such a simplex or cube with another simplex, cube or some $D(n, \frac{1}{2})$ to obtain the E-polytope of this product.

4   Properties of the family E_{mn}

This section collects several properties of the polytopes E_{mn}. In particular we count degrees of freedom for the realisation of E_{33} and prove that not all combinatorial symmetries of E_{mn} are geometrically realisable.

4.1 Self-Duality

The polytopes $C_m \times C_n$ are simple, thus we know from [PZ04, Thm. 1.6] that E_{mn} is 2-simple and 2-simplicial (that is, all 2-faces are triangles and all edges are in 3 facets). In particular the f-vector of E_{mn} is symmetric (cf. Eq. (1)).

The polytopes E_{mn} are in fact self-dual. This is not true for arbitrary 2-simple, 2-simplicial polytopes, which can be seen from the hypersimplex $E(\Delta)$ obtained from the 4-simplex Δ. This polytope has a facet-transitive automorphism group acting on its 10 bipyramidal facets, while the dual has 5 tetrahedral and 5 octahedral facets.

Theorem 4.1 (Ziegler [Zie03]). *Each of the polytopes E_{mn} ($n, m \geq 3$) is self-dual, with an anti-automorphism of order 2.*

Proof. Number the vertices of an k-gon C_k consecutively by $v_0, \ldots, v_{k-1}$. We take indices modulo k. The vertices of $C_m \times C_n$ are $v_{i,j} := (v_i, v_j)$ for $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$. We have two types of facets in the product:

$$F_i' = \text{conv}(\{v_{ij}, v_{i+1,j} \mid j = 0, \ldots, n-1\})$$
$$F_j'' = \text{conv}(\{v_{ij}, v_{i,j+1} \mid i = 0, \ldots, m-1\}).$$

We denote the new vertex beyond F_i' by v_i' and the one beyond F_i'' by v_i''. The facets of $E(C_m \times C_n)$ are now of the form

$$G_{ij} = \text{conv}(v_{ij}, v_{i+1,j}, v_{i,j+1}, v_{i+1,j+1}, v_i', v_j'')$$
$$G_i' = \text{conv}(\{v_{ij} \mid j = 0, \ldots, n-1\}, v_{i-1}', v_i')$$
$$G_j'' = \text{conv}(\{v_{ij} \mid i = 0, \ldots, m-1\}, v_{j-1}'', v_j'').$$
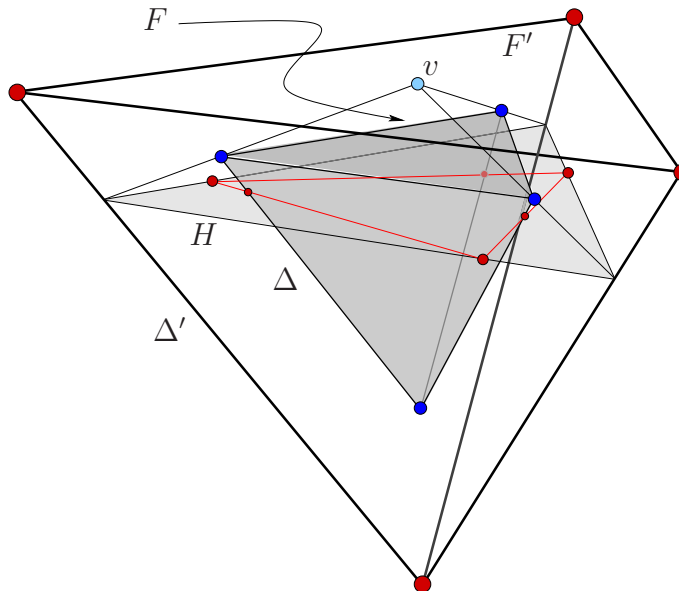
Fig. 3.3: The construction for a 3-simplex. The solution for $d = 2$ used in the plane H is indicated with thin lines.

From this we can read off the facets a vertex is contained in:

$$v_{ij} \in G_{ij}, G_{i-1,j}, G_{i,j-1}, G_{i-1,j-1}, G'_{i-1}, G''_{j-1}$$
$$v'_i \in G'_i, G'_{i+1}, G_{ij} \text{ for } j = 0, \ldots, n-1$$
$$v''_j \in G''_j, G''_{j+1}, G_{ij} \text{ for } i = 0, \ldots, m-1.$$

Thus the following correspondence gives a self-duality of order 2:

$$G_{ij} \longleftrightarrow v_{-i,-j} \qquad G'_i \longleftrightarrow v'_{-i} \qquad G''_j \longleftrightarrow v''_{-j} \qquad \square$$

For $m = n$, this result was obtained previously by Gévay [Gev04].

Remark 4.2. There are examples of 3-polytopes that are self-dual, but that do not have a self-duality of order 2 (cf. [Grü03, Ex. 3.4.3, p.52d]).

4.2 E_{mn} Constructed from Regular Polygons

Only in a few cases there are "more symmetric" realisations of the polytopes E_{mn}: We prove that there are only five choices of pairs (m, n) (up to interchanging m and n) such that we can take regular polygons as input for the construction described in Theorem 2.1 in the restricted version of Corollary 2.2. We will see in the next section that these five cases are also the only cases in which the product of two cyclic groups induced from rotation of the vertices in the two factors can be a subgroup of the geometric symmetry group. The next theorem is based on Santos [San00, Rem. 13], [San03].

Theorem 4.3. *There are polytopal realisations of E_{mn} for which projection onto the first and last two coordinates yields in both cases*
(1) *regular polygons for the polygon in $C_m \times C_n$ and its E-construction,*
(2) *and all intersection ratios are equal in each factor*
if and only if $\frac{1}{m} + \frac{1}{n} \geq \frac{1}{2}$.

Proof. The condition on the ratio implies that the images of the maps β_0 and β_1 appearing in the construction of E_{mn} are single points in the interior of the polygons C_m and C_n. These points must be the barycentres if the polygons are regular. We may assume that this is the origin.
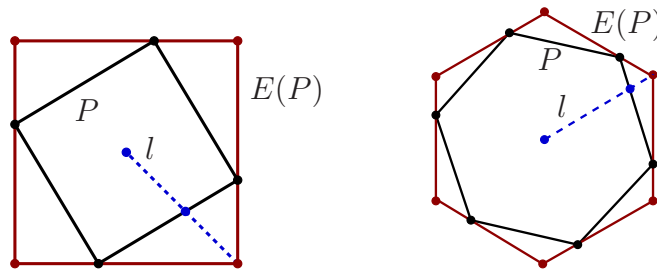

Fig. 4.1: Two projections that satisfy the restrictions of Theorem 4.3

We can now generate all configurations of a regular polygon C_m together with $E(C_m)$ in the following way: Start with a regular polygon $E(C_m)$ centred at the origin and choose a vertex for C_m on each of the edges. As C_m is regular, the vertices of C_m divide each edge with equal ratio. The segments considered in (B) are the segments l between the origin and a vertex of $E(C_m)$. We are interested in the possible values of the ratio with which they are intersected by the edges of C_m.

Choosing the vertices of C_m close to those of $E(C_m)$ we see that we can have an arbitrarily high portion of l inside C_m. On the other hand, the portion inside C_m is minimised when we place the vertices of C_m in the centre of the edges. In this case, the fraction of l outside C_m is $\sin^2(\frac{\pi}{n})$. By condition (B), the fraction lying outside for one polygon and its E-construction has to match the fraction lying inside for the other polygon. This gives the following inequalities:

$$1 - \sin^2\left(\frac{\pi}{m}\right) \leq \sin^2\left(\frac{\pi}{n}\right) \qquad \text{and} \qquad 1 - \sin^2\left(\frac{\pi}{n}\right) \leq \sin^2\left(\frac{\pi}{m}\right),$$

which are equivalent to the condition given in the theorem. $\qquad\square$

We can determine all possible values for the inequalities of the theorem explicitly.

Corollary 4.4. *There are realisations of E_{mn} from regular polytopes only for the following pairs (m, n) (up to interchanging m and n):*

$$(3,3) \, , \; (3,4) \, , \; (3,5) \, , \; (3,6) \, , \; (4,4) \qquad\qquad \square$$

Remark 4.5. We made assumption (2) in Theorem 4.3 mainly because this is the case we need in the next section. A less restrictive version of "symmetry" would only require the points in the images of β_0 and β_1 to also form regular polygons (if we take the points in the order induced by the E-construction of the other factor). For small $m = n$ this has solutions where all points in the images are different. See Table 4.2 for an example of an E_{44}. Note however, that this severely reduces the number of geometric symmetries compared to the case of the theorem.

4.3 Combinatorial versus Geometric Symmetries

There are two different notions of symmetry for a polytope P.

Definition 4.6. *Let P be a polytope with a given geometric realisation. Any affine transformation T of the ambient space that preserves P set-wise is called a* geometric symmetry transformation. *The group of all such transformations is called the* geometric symmetry group.

To any polytope P we can associate the poset of all faces of P ordered by inclusion. This is called the face lattice $F(P)$ *of P. A combinatorial symmetry of P is an automorphism of $F(P)$. The group of all combinatorial symmetries is the* combinatorial symmetry group.

The combinatorial symmetry group is independent of a realisation, while the geometric symmetry group highly depends on the choice of the realisation.

The three symmetries involved in the proof of Theorem 4.7

Notation:

- $\triangleright$ v_0, v_1, v_2: vertices of C_3
- $\triangleright$ w_0, w_1, w_2, w_3 vertices of C_4
- $\triangleright$ $e(j)$: edge from vertex number j to $j + 1$ (mod 3 or 4) in both polygons.

Number the vertices p_k of P_{34} in the following way:

$0 \leq k \leq 11$: vertices $(v_{k \text{ div } 4}, w_{k - 4(k \text{ div } 4)})$
$12 \leq k \leq 14$: vertices added above $e(k - 12) \times C_4$
$15 \leq k \leq 19$: vertices added above $C_3 \times e(k - 15)$

Then the combinatorial symmetries are given as (permutation notation, vertex numbers of p_k):

$$\tilde{S}_3 := (0, 4, 8)(1, 5, 9)(2, 6, 10)(3, 7, 11)(12, 13, 14)(15)(16)(17)(18)$$
$$\tilde{S}_4 := (0, 1, 2, 3)(4, 5, 6, 7)(8, 9, 10, 11)(12)(13)(14)(15, 16, 17, 18)$$
$$T := (0, 5, 10, 3, 4, 9, 2, 7, 8, 1, 6, 11)(12, 13, 14)(15, 16, 17, 18)$$

Table 4.1: The combinatorial symmetries $\tilde{S}_3$, $\tilde{S}_4$, and T acting on P_{34}.

A geometric symmetry maps k-faces to k-faces and preserves incidences. Therefore any geometric symmetry induces a combinatorial symmetry. On the other hand, a combinatorial symmetry in general does not induce a geometric one. However, there are not many examples known of polytopes where these two groups differ for all possible geometric realisations of a polytope. Bokowski, Ewald, and Kleinschmidt have provided a 4-dimensional example on 10 vertices in [BEK84]. Dimension 4 is smallest possible for such examples, as it is known, that for 3-polytopes, and for polytopes with few vertices in any dimension, there are realisations for which geometric and combinatorial symmetry group coincide (see [Man71] for the first and [Grü03, p.120] for the second result). We show that our product construction provides an infinite series of 4-polytopes with non-realisable geometric symmetries. We construct an explicit example of such a symmetry for the proof. Previously it was observed by Gévay that there is no polytopal realisation of the CW spheres E_{mm} with the full symmetry group, except in the case $m = 4$. This is also a consequence of Corollary 4.8 below.

Theorem 4.7. *For relatively prime $m, n \geq 5$ all E_{mn} admit combinatorial symmetries that cannot be realised as affine symmetry transformations of a geometric realisation of E_{mn}.*

Proof. Let P_{mn} be any geometric realisation of a polytope combinatorially equivalent to an E_{mn}. Seen as a PL sphere, P_{mn} can still be viewed as the result of the E-construction applied to a PL sphere which is combinatorially equivalent to a product of two polygons.

Here is a non-realisable combinatorial symmetry T of P_{mn}. Let C_m and C_n denote polygons with vertices $v_0, \ldots, v_{m-1}$ resp. $w_0, \ldots, w_{n-1}$ numbered in cyclic order. We take indices modulo m resp. n. Let S be the combinatorial symmetry of a polygon that maps the j-th to the $(j+1)$-th vertex.

S induces a combinatorial symmetry S_m on $C_m \times C_n$ by mapping a vertex (v_i, w_j) to (v_{i+1}, w_j) for any $0 \leq j \leq m-1$. Similarly S induces a symmetry S_n shifting the vertices of C_n. Both symmetries uniquely extend to combinatorial symmetries $\tilde{S}_m$ and $\tilde{S}_n$ of $E(C_m \times C_n)$. Let T be the combinatorial symmetry of P_{mn} obtained by first applying $\tilde{S}_m$ and then $\tilde{S}_n$. See Table 4.1 for an example.

A geometric realisation of P_{mn} need not have the product structure induced by the construction of Theorem 2.1. However, by looking at vertex degrees, for $m, n \geq 5$ we can decide which of the vertices of P_{mn} "belong" to the product and which are "added" by the E-construction: A vertex of the product always has degree 8, as $C_m \times C_n$ is simple, so any vertex has four neighbours and is in four facets. The added vertices all have degree $2m$ or $2n$. Denote the vertex sets by V_p and V_e.

The proof is roughly as follows. Suppose there is a geometric realisation T_g of T for some P_{mn}. First we prove that any P_{mn} having T_g as a geometric symmetry has the form of the construction in Theorem 2.1. Then the symmetry implies that both factors are of the form defined in Theorem 4.3. Corollary 4.4 finally tells us that for $m, n \geq 5$ there are no such realisations.

As T_g set-wise fixes the the vertices of P_{mn} it also fixes the centroid of the vertices of P_{mn}. After a suitable translation we can assume that T_g is a linear transformation. As m and n are relatively prime, there is a $k_m \in \mathbb{N}$ such that $T_m := T_g^{k_m}$ restricted to the set V_p acts as $\tilde{S}_m$. Similarly there is a k_n such that $T_n := T_g^{k_n}$ reduces to a realisation of $\tilde{S}_n$. T_m and T_n are linear transformations.

By construction P_{mn} has two different combinatorial types of facets: Bipyramids over an m-gon and over an n-gon. For any facet we call the vertices of the polygon (i.e. those vertices of the facet belonging to V_p) the base vertices.

Let F be a facet of P_{mn} of the first type. The symmetry T_m shifts the base vertices by one and fixes the two apices. Thus, T_m also fixes the centroid c_F of the base vertices of F and T_m restricted to the hyperplane H_F defined by F is a linear transformation T_m^F in H_F (if we put the origin of H_F in c_F). Now T_m fixes the two apices of F and thus fixes the whole line through the apices. So T_m^F splits into a map fixing the axis and a linear transformation of a two dimensional transversal subspace. The axis must contain c_F and the subspace the base vertices of F. So the base vertices of F lie in a common two dimensional affine subspace of $\mathbb{R}^4$. Similarly, the base vertices of any other bipyramidal facet with a base equivalent to C_m lie in a common 2-plane. These 2-planes are set-wise preserved by T_m and therefore must be parallel.

The same argument proves that all bases of facets combinatorially being bipyramids over n-gons do lie in parallel 2-planes. These 2-planes must be transversal to the 2-planes containing the m-gons: Otherwise the vertices in V_p all lie in a three dimensional subspace. As P_{mn} is four dimensional, at least one of the vertices of V_e has to lie outside this 3-space. But there are no edges between vertices in V_e.

Applying an appropriate linear transformation to P_{mn} we can assume that the 2-spaces containing the m-gons are parallel to the x_1-x_2-plane and the ones containing the C_n are parallel to the x_3-x_4-plane. T rotates the copies of C_m in P_{mn}, so they must all be equivalent. Similarly, all the polygons C_n are equivalent. So P_{mn} is an instance of Theorem 2.1.

Consider again the facet F with base equivalent to C_m and the restricted map T_m^F. Further restricting T_m^F to the subspace containing the base vertices defines a linear map T_b on $\mathbb{R}^2$ shifting the vertices of a polygon. So T_b generates a finite subgroup of $Gl(2, \mathbb{R})$ and therefore must be conjugate to an element of $O(2, \mathbb{R})$ (cf. [Sch11], see also [McM68]). The same argument applies to facets with base C_n. As the copies of C_m and C_n lie in transversal subspaces of $\mathbb{R}^4$, we can apply the conjugation for C_m and C_n simultaneously and therefore both polygons are regular up to an affine map.

Finally look at the n vertices added above facets of P_{mn} of the type $C_m \times F$ for an edge F of C_n. Projecting onto the 2-space of C_m they lie inside C_m (they form the set S_1 in the construction of Theorem 2.1). They are fixed by the symmetry $\tilde{S}_m$. As this map has only one fixed point, the points in S_1 must coincide. The same applies to the added vertices above facets of type $F \times C_n$. (Note that, even though T is a symmetry of the E_{44} in Table 4.2, the map $\tilde{S}_4$ is not, and cannot be obtained as a power of T.)

Now we are in the situation described in Section 4.2. But according to Corollary 4.4 this can only be the case if at least one of m and n is less than 5. This proves Theorem 4.7. $\qquad\square$

The same argument also proves that Corollary 4.4 describes all possible cases for which P_{mn} can have the product $\mathbb{Z}_m \times \mathbb{Z}_n$ of two cyclic groups induced by the rotation of the vertices in the two polygons as a subgroup of its geometric symmetry group. In this case we do not need m and n to be relatively prime as the two symmetries $\tilde{S}_m$ and $\tilde{S}_n$ itself are contained in $\mathbb{Z}_m \times \mathbb{Z}_n$ acting on P_{mn}.

Corollary 4.8. *The combinatorial symmetry group of E_{mn} contains a subgroup G isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$ induced by rotation in the two polygon factors.*

The geometric symmetry group of a polytope P_{mn} combinatorially equivalent to E_{mn} can contain a subgroup inducing G on the face lattice only for $(m, n) = \{(3, 3), (3, 4), (3, 5), (3, 6), (4, 4)\}$ (up to interchanging m and n). $\qquad\square$

Hence, in particular, E_{44} and E_{36} are the only two polytopes that have a geometric realisation realising all combinatorial symmetries.

Remark 4.9. Gévay [Gev04] pointed out that along the lines of Theorem 4.7 one can also prove that the only "perfect" polytopes among the realisations of the E_{mn} are the regular 24-cell and E_{33} constructed as in Theorem 4.3 with intersection ratio $r = 1/2$. A rough definition of perfectness is as follows: A geometric realisation P of a polytope is *perfect* if all other geometric realisations having, up to conjugation with an isometry, the same subset of the affine transformations as symmetry group, are already isometric to P. See [Gev02] for a precise definition.

4.4 Realisation spaces of E_{33} and E_{44}

We determine the degrees of freedom that we have in the choice of coordinates for E_{33}. We consider two realisations to be equal if they only differ by a projective transformation. Thus, we will be interested in the dimension of the following spaces.

Definition 4.10. *The* realisation space *of a d-polytope P with n vertices is the space $\mathcal{R}(P)$ of all sets of n points in R^d whose convex hull is combinatorially equivalent to P. $\mathcal{R}(P)$ is a subset of $\mathbb{R}^{d \cdot n}$.*

The projective realisation space *$\mathcal{R}_{proj}(P)$ of a polytope is the space of all possible geometric realisations of a polytope, up to projective equivalence. It is the quotient space of $\mathcal{R}(P)$ where two realisations are equivalent if there is a projective transformation mapping one onto the other.*

We work out the case of E_{33} explicitly and present a simple 4-parameter family of $E_{44}s$. We prove that this family intersects four different equivalence classes of the projective realisation space $\mathcal{R}_{proj}(E_{44})$. Therefore, this space is at least four dimensional.

THE REALISATION SPACE OF E_{33}. The vertex sets of all realisations of E_{33} obtained from the construction in Theorem 2.1 contain the vertex set of an orthogonal product $C_3 \times C_3$ of two triangles. This reduces the number of possible degrees of freedom compared to an arbitrary realisation. The next theorem determines the dimension of the space of all realisations of E_{33} that are projectively equivalent to a realisation containing such an orthogonal product.

Theorem 4.11. $\dim(\mathcal{R}_{proj}(E_{33})) \geq 9$.

Before we prove this we introduce a special way to construct realisations of two triangles and their E-polytopes satisfying the conditions (A) and (B).

Theorem 4.12. *Given two (arbitrary) triangles Δ and Δ' there is an open subset R in $\mathbb{R}^9$ such that, if we take the nine entries of a vector in that set as the nine ratios appearing in* (B) *(in some previously fixed order), then there is a realisation of E_{33} having these intersection ratios.*

Proof. This is basically proven by describing a realisation as a solution of a set of linear equations, but we have to introduce some notation to write down the equations.

In the following let the index x always run through $\{a, b, c\}$ and and y through $\{a', b', c'\}$. Fix two triangles Δ and Δ' and let s_a, s_b, s_c be the sides of Δ and $s_{a'}, s_{b'}, s_{c'}$ the sides of Δ'. By a translation in each of the two factors we can assume that they both contain the origin. Denote the nine ratios by r_{xy} for $x \in \{a, b, c\}$ and $y \in \{a', b', c'\}$. See Figure 4.2. To simplify the notation we introduce the parameters $R_{xy} := \frac{r_{xy}}{1 - r_{xy}}$.

Let g_x be a line outside Δ parallel to s_x at a distance δ_x. These three lines will afterwards contain the vertices of $E(\Delta)$, which is a triangle containing the vertices of Δ in its edges. Similarly, define the line g_y at distance δ_y from s_y for Δ'.

Let l_{ay} define a line parallel to s_a lying on the other side of a as g_a at distance $R_{ay}\delta_a$ from s_a. Similarly define the lines l_{by} and l_{cy} parallel to b and c. Thus, any segment starting on g_x and ending on l_{xy} is divided by s_x with a ratio of r_{xy}. For the triangle Δ' we define lines l'_{ay} at distance $1/R_{ay}$ parallel to s_y and on the other side as g_y. Finally, define (outward pointing) normal vectors n_x and n_y and levels λ_x, and λ_y such that points $u \in s_x$ satisfy $\langle n_x, u\rangle - \lambda_x = 0$ and points $v \in s_y$ satisfy $\langle n_y, v\rangle - \lambda_y = 0$.

Consider now e.g. the ratio $r_{ab'}$. Choose a vertex v_a of $E(\Delta)$ on g_a, a point w_a on the line $l'_{ab'}$ and in the interior of Δ', a vertex $v'_{b'}$ of $E(\Delta')$ lying on $g_{b'}$ and a point $w'_{b'}$ in the interior of Δ on the line $l_{ab'}$. The points w_a and $w'_{b'}$ will become the corresponding points to v_a and $v'_{b'}$ under the maps β_0 and β_1 of (B). The part of the segment $h_{ab'}$ between v_a and $w'_{b'}$ lying inside Δ has length $r_{ab'}|h_{ab'}|$ (where $|h|$ denotes the length of a segment h) and the part of the segment $h_{b'a}$ between $v'_{b'}$ and w_a inside Δ' has length $(1 - r_{ab'})|h_{b'a}|$. So the condition set by the ratio $r_{ab'}$ will be satisfied by this choice of w_a and $w'_{b'}$.

To satisfy all conditions on the ratios that involve w_a, we have to choose w_a such that it lies as well on the lines l_{ay} and in the interior of Δ. Similar conditions hold for the two other points inside Δ and for the three points inside Δ'. Therefore, finding a feasible solution amounts to finding a solution to the following set of 18 linear equations and six linear inequalities.

$$\lambda_x = \langle n_x, w'_y\rangle + R_{xy}\delta_x$$
$$\lambda_y = \langle n_y, w_x\rangle + 1/R_{xy}\delta_y$$
$$0 < \delta_x, \delta_y$$

for all $x \in \{a, b, c\}$ and $y \in \{a', b', c'\}$. Here the coordinates of the points w_x, w'_y and the distances δ_x, δ_y are the free variables, and the ratios are the parameters. The first and the second set of equations are connected via the ratios. As the equations and inequalities depend smoothly on the nine parameters, it suffices for
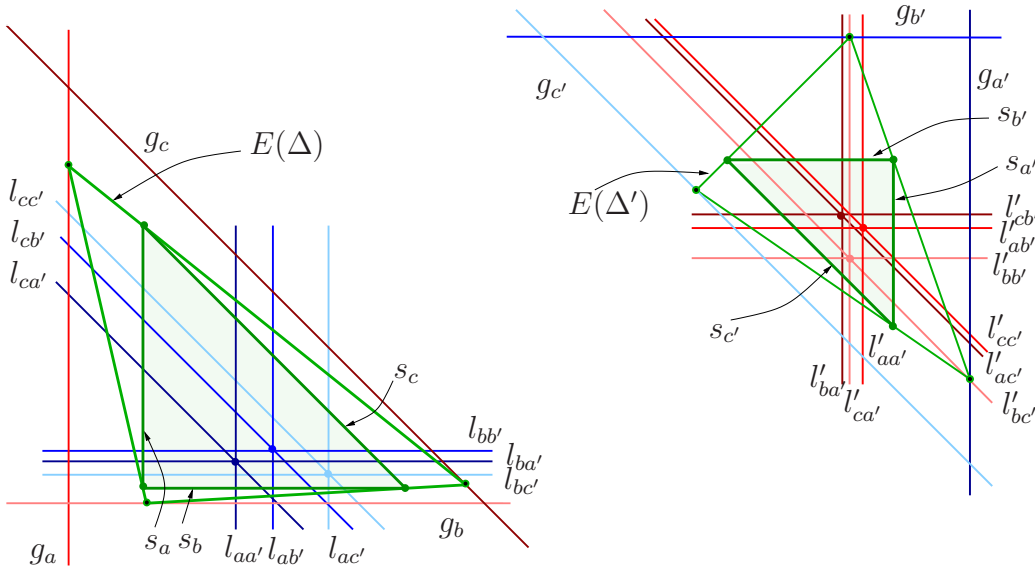

Fig. 4.2: Construction of the triangles

the proof of the theorem to show that there exists at least one feasible solution of this system. Such a solution is shown in Figure 4.2 and in Table 4.3 (for some fixed product of two triangles, but this can be projectively transformed to any other).

To finally obtain $E(\Delta)$ we have to choose vertices on the lines g_a, g_b, and g_c such that the edges contain the vertices of Δ. Unless the distances δ_a, δ_b, and δ_c are too large compared to the size of Δ, there are always two solutions to this problem (one of the solutions for the E_{33} mentioned above is given in Table 4.3, the other is obtained by reflection), which depend continuously on the distances δ_x, δ_y (There is no solution otherwise). Similarly we can construct $E(\Delta')$. $\qquad\square$

From this construction method the proof of Theorem 4.11 is straightforward:

Proof of Theorem 4.11. All triangles in $\mathbb{R}^2$ are projectively equivalent. Therefore, up to projective equivalence, there is only one geometric realisation of an orthogonal product of two triangles. Thus, in the following we can fix our preferred orthogonal product of two triangles and count the degrees of freedom for adding the remaining vertices without having to worry about projective equivalence anymore. But according to Theorem 4.12 we have, for any choice of two triangles, nine degrees of freedom for the choice of the remaining vertices. $\qquad\square$

Remark 4.13. There might still be geometric realisations of a polytope combinatorially equivalent to E_{33} that are not projectively equivalent to a polytope

$$
\begin{array}{rrrr}
[& 1 & 1 & 1 & 1] \\
[& 1 & 1 & 1 & -1] \\
[& 1 & 1 & -1 & 1] \\
[& 1 & 1 & -1 & -1] \\
[& 1 & -1 & 1 & 1] \\
[& 1 & -1 & 1 & -1] \\
[& 1 & -1 & -1 & 1] \\
[& 1 & -1 & -1 & -1] \\
[& -1 & 1 & 1 & 1] \\
[& -1 & 1 & 1 & -1] \\
[& -1 & 1 & -1 & 1] \\
[& -1 & 1 & -1 & -1] \\
[& -1 & -1 & 1 & 1] \\
[& -1 & -1 & 1 & -1] \\
[& -1 & -1 & -1 & 1] \\
[& -1 & -1 & -1 & -1] \\
[& 3/5 & 9/5 & -3/5 & -3/5] \\
[& 9/5 & -3/5 & -3/5 & 3/5] \\
[& -3/5 & -9/5 & 3/5 & 3/5] \\
[& -9/5 & 3/5 & 3/5 & -3/5] \\
[& -3/5 & 3/5 & 3/5 & 9/5] \\
[& -3/5 & -3/5 & -9/5 & 3/5] \\
[& 3/5 & -3/5 & -3/5 & -9/5] \\
[& 3/5 & 3/5 & 9/5 & -3/5]
\end{array}
$$

$$
\begin{array}{rrrr}
[& 1 & 0 & 1 & 0] \\
[& 1 & 0 & 0 & 0] \\
[& 1 & 0 & 0 & 1] \\
[& 0 & 0 & 1 & 0] \\
[& 0 & 0 & 0 & 0] \\
[& 0 & 0 & 0 & 1] \\
[& 0 & 1 & 1 & 0] \\
[& 0 & 1 & 0 & 0] \\
[& 0 & 1 & 0 & 1] \\
[& 9/247 & 289/247 & 819/1387 & 364/1387] \\
[& 289/247 & -51/247 & 364/1387 & 204/1387] \\
[& -51/247 & 9/247 & 204/1387 & 819/1387] \\
[& 153/494 & 34/247 & 169/1387 & 1764/1387] \\
[& 34/247 & 21/38 & 1764/1387 & -546/1387] \\
[& 21/38 & 153/494 & -546/1387 & 169/1387]
\end{array}
$$

Table 4.2: An E_{44} from regular squares, but not satisfying (2) of Theorem 4.3

Table 4.3: The coordinates of a feasible non-degenerate solution. See Figure 2.1 for a drawing of the two factors
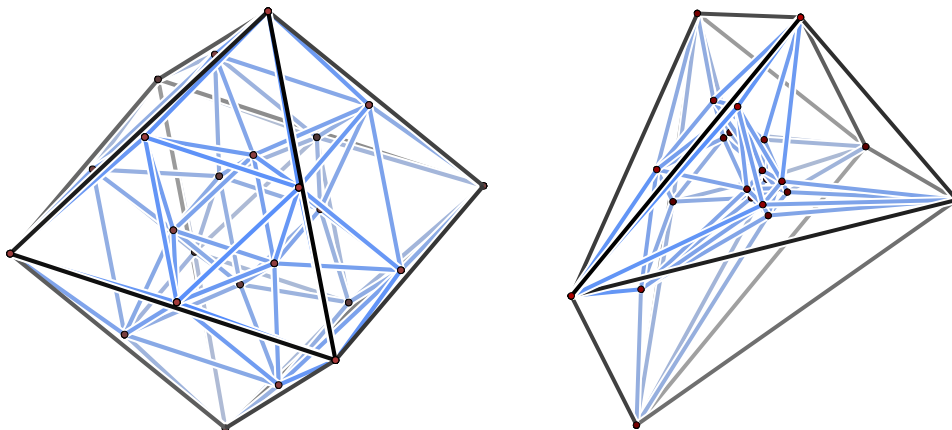
Fig. 4.3: The regular 24-cell and one other realisation from the family of Table 4.4

containing an orthogonal product of two triangles. Thus, a priori Theorem 4.11 describes only a subset of the whole realisation space $\mathcal{R}_{proj}(E_{33})$.

THE 24-CELL. From our method to realise the E-construction of products of polygons we can obtain new geometric realisations of the 24-cell.

A 4-Parameter-Family of 24-Cells. For $m, n > 3$ we cannot determine the degrees of freedom in the above way anymore. Taking the mn ratios as input we obtain $2mn$ equations and $m + n$ inequalities for only $3(m + n)$ variables. This is not merely a problem of the method. There are in fact additional restrictions on a realisation, as the lengths of the segments from an interior point to the vertices of the E-construction cannot be viewed as independent variables anymore (consider e.g. a square and a pair of opposite vertices of its E-construction). However, also for the 24-cell it is not difficult to construct projectively non-equivalent geometric realisations.

Table 4.4 shows a simple example of a 4-parameter family of 24-cells, where all four parameters range between -1 and 1. This family spans a 4-dimensional subset of the projective realisation space, which can be seen in the following way. The vertex set of the regular 24-cell contains the vertex set of three different standard cubes: If you set all parameters to 0 then (in the order given in Table 4.4) the first sixteen, the last sixteen and the first and last eighth vertices each form a standard cube. Their 2-faces (squares) are not anymore present in the 24-cell, but they still lie on a codimension-2-subspace, which is preserved by any projective transformation (e.g. vertices $15, 16, 17, 18$ in Table 4.4). Letting the parameters diverge from 0 destroys some of these "internal" squares, necessarily resulting in projectively different 24-cells. This can also be seen in the Schlegel diagrams in Figure 4.3: Observe the three squares contained in the octahedral face on which the polytope is projected.

Remark 4.14. Clearly, not all possible realisations of the 24-cell are contained in this 4-parameter family. The 24-cell in Table 4.5 is also a result of the construction and has no projective automorphisms.

4.5 Fatness of Polytopes

The classification of f- and flag vectors for polytopes in dimension $d \geq 4$ is an important unsolved problem in polytope theory. See [Bay87], [Zie02] for some background on this problem and overviews of the known results.

Ziegler [Zie02] proposed to look at the following quantity (called the "fatness" of a polytope P) on the entries of these two vectors.

$$F(P) := \frac{f_1 + f_2 - 20}{f_0 - f_3 - 10}$$

where (f_0, f_1, f_2, f_3) is the f-vector of any 4-polytope different from the simplex (in [EKZ03] there is a slightly different definition). The fatness of polytopes produced from the E-construction applied to simple 4-polytopes is bounded by 6, cf. [EKZ03, p. 3]. Eppstein, Kuperberg, and Ziegler provided a polytope Q resulting from a gluing of 600-cells that has fatness around 5.073 in the definition of [Zie02] (The E-construction also works for some non-simple polytopes, but all known examples don't have a higher fatness). They also showed that for regular CW 3-spheres fatness is unbounded. But they neither found polytopes with fatness higher than 5.073 nor an upper bound on fatness for arbitrary polytopes.

For our family E_{mn} we get according to the f-vector computation in (1):

$$F(E(C_m \times C_n)) = \frac{12mn - 20}{2mn + 2m + 2n - 10} \longrightarrow 6$$

$[\ -1$	-1	-1	$-1]$
$[\ 1$	1	-1	$-1]$
$[\ 1$	-1	1	$-1]$
$[\ -1$	1	1	$-1]$
$[\ 1$	-1	-1	$1]$
$[\ -1$	1	-1	$1]$
$[\ -1$	-1	1	$1]$
$[\ 1$	1	1	$1]$
$[\ 1$	-1	-1	$-1]$
$[\ -1$	1	-1	$-1]$
$[\ -1$	-1	1	$-1]$
$[\ -1$	-1	-1	$1]$
$[\ 1$	1	1	$-1]$
$[\ 1$	1	-1	$1]$
$[\ 1$	-1	1	$1]$
$[\ -1$	1	1	$1]$
$[\ a_1$	b_1	a_2	$-2-b_2]$
$[\ a_1$	b_1	$2-a_2$	$b_2]$
$[\ a_1$	b_1	a_2	$2-b_2]$
$[\ a_1$	b_1	$-2-a_2$	$b_2]$
$[\ a_1$	$2-b_1$	a_2	$b_2]$
$[-2-a_1$	b_1	a_2	$b_2]$
$[\ a_1$	$-2-b_1$	a_2	$b_2]$
$[\ 2-a_1$	b_1	a_2	$b_2]$

$[\ -1$	$5/4$	-1	$1],$
$[\ -1$	$5/4$	-1	$-1],$
$[\ -1$	$5/4$	1	$-1],$
$[\ -1$	$5/4$	$5/3$	$1],$
$[\ -1$	-1	-1	$1],$
$[\ -1$	-1	-1	$-1],$
$[\ -1$	-1	1	$-1],$
$[\ -1$	-1	$5/3$	$1],$
$[\ 1$	-1	-1	$1],$
$[\ 1$	-1	-1	$-1],$
$[\ 1$	-1	1	$-1],$
$[\ 1$	-1	$5/3$	$1],$
$[\ 1$	$23/12$	-1	$1],$
$[\ 1$	$23/12$	-1	$-1],$
$[\ 1$	$23/12$	1	$-1],$
$[\ 1$	$23/12$	$5/3$	$1],$
$[-1/2$	$-1/2$	$-3/2$	$1/2],$
$[-1/2$	$-1/2$	$-5/6$	$-3/2],$
$[-1/2$	$-1/2$	$17/6$	$-1/2],$
$[-1/2$	$-1/2$	$1/2$	$5/2],$
$[-3/2$	$-5/6$	$-1/2$	$-1/2],$
$[\ 1/2$	$-3/2$	$-1/2$	$-1/2],$
$[\ 5/2$	$1/2$	$-1/2$	$-1/2],$
$[-1/2$	$10/3$	$-1/2$	$-1/2],$

Table 4.4: Vertices of a family of 24-cells

Table 4.5: A 24-cell without any projective automorphisms

for $m, n \to \infty$. Thus for $m, n \geq 10$ our polytopes are "fatter" than the above mentioned example from [EKZ03]. As products of polygons are simple, our family of polytopes is "best possible" within this setting. However, Ziegler recently has constructed a class of polytopes (not "E-polytopes") with fatness arbitrarily close to 9 by considering projections of products of polygons to $\mathbb{R}^4$ [Zie04].

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